

Acta Math. Sin. New Ser.  
2008, 24(5), pp. 705–736

## SPECTRAL GAP AND LOGARITHMIC SOBOLEV CONSTANT FOR CONTINUOUS SPIN SYSTEMS

MU-FA CHEN

(School of Mathematical Sciences, Beijing  
Normal University, Beijing 100875, P.R. China)  
E-mail: mfchen@bnu.edu.cn  
September 19, 2007

**ABSTRACT.** The aim of this paper is to study the spectral gap and the logarithmic Sobolev constant for continuous spin systems. A simple but general result for estimating the spectral gap of finite dimensional systems is given by Theorem 1.1, in terms of the spectral gap for one-dimensional marginals. The study of the topic provides us a chance, and it is indeed another aim of the paper, to justify the power of the results obtained previously. The exact order in dimension one (Proposition 1.4), and then the precise leading order and the explicit positive regions of the spectral gap and the logarithmic Sobolev constant for two typical infinite-dimensional models are presented (Theorems 6.2 and 6.3). Since we are interested in explicit estimates, the computations become quite involved. A long section (Section 4) is devoted to the study of the spectral gap in dimension one.

**1. Introduction.** The local Poincaré inequalities (equivalently, spectral gaps) and logarithmic Sobolev inequalities for unbounded continuous spin systems have recently obtained a lot of attention by many authors [1]–[11]. For the present status of the study and further references, the readers may refer to the comprehensive survey article [7]. In the most of the publications, the authors consider mainly the perturbation regime with convex phase at infinity. More recently, the non-convex phase is treated for a class of spin systems based on a criterion for the weighted Hardy inequalities.

The main purpose of this paper is to propose a general formula for the local spectral gaps of continuous spin systems. Let us start from finite dimensions. Let

---

1991 *Mathematics Subject Classification.* 60K35.

*Key words and phrases.* Spectral gap, logarithmic Sobolev constant, spin system, principle eigenvalue.

Research supported in part by the Creative Research Group Fund of the National Natural Science Foundation of China (No. 10121101) and by the “985” project from the Ministry of Education in China.

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

$U \in C^\infty(\mathbb{R}^n)$  satisfy  $Z := \int_{\mathbb{R}^n} e^{-U} dx < \infty$  and set  $d\mu_U = e^{-U} dx / Z$ . Throughout the paper, we use a particular notation  $x_{\setminus i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ , obtained from  $x := (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  by removing the  $i$ th component. Clearly, the conditional distribution of  $x_i$  given  $x_{\setminus i}$  under  $\mu_U$  is as follows:

$$\mu_U^{x_{\setminus i}}(dx_i) = e^{-U} dx_i / Z(x_{\setminus i}), \quad (1.1)$$

where  $Z(x_{\setminus i}) = \int_{\mathbb{R}} e^{-U(x)} dx_i$ . The measure  $\mu_U^{x_{\setminus i}}$  is the invariant probability measure of the one-dimensional diffusion process, corresponding to the operator  $L_i^{x_{\setminus i}} = d^2/dx_i^2 - \partial_i U d/dx_i$ .

Let  $L = \Delta - \langle \nabla U, \nabla \rangle$ . Recall that the spectral gap  $\lambda_1(L) = \lambda_1(U)$  is the largest constant  $\kappa$  in the following Poincaré inequality

$$\kappa \text{Var}_{\mu_U}(f) \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\mu_U =: D(f), \quad f \in C_0^\infty(\mathbb{R}^n), \quad (1.2)$$

where  $\text{Var}_{\mu_U}(f)$  is the variation of  $f$  with respect to  $\mu_U$  and  $C_0^\infty(\mathbb{R}^n)$  is the set of smooth functions with compact supports.

Denote by  $\lambda_1^{x_{\setminus i}} = \lambda_1(L_i^{x_{\setminus i}})$  the spectral gap of the one-dimensional operator  $L_i^{x_{\setminus i}}$ :

$$\lambda_1^{x_{\setminus i}} \text{Var}_{\mu_U^{x_{\setminus i}}}(f) \leq \int_{\mathbb{R}} f'^2 d\mu_U^{x_{\setminus i}}, \quad f \in C_0^\infty(\mathbb{R}). \quad (1.3)$$

Then, we can state our variational formula for the lower bounds of  $\lambda_1(U)$  as follows.

**Theorem 1.1.** *Define*

$$(\widetilde{\text{Hess}}(U))_{ij} = \begin{cases} \lambda_1^{x_{\setminus i}}, & \text{if } i = j \\ \partial_{ij} U, & \text{if } i \neq j, \end{cases}$$

where  $(\text{Hess}(U))_{ij} = \partial_{ij} U := \partial^2 U / \partial x_i \partial x_j$ . Then we have

$$\begin{aligned} \lambda_1(U) &\geq \inf_{x \in \mathbb{R}^n} \lambda_{\min}(\widetilde{\text{Hess}}(U)(x)) \\ &\geq \inf_{x \in \mathbb{R}^n} \sup_w \min_{1 \leq i \leq n} \left( \lambda_1^{x_{\setminus i}} - \sum_{j:j \neq i} |\partial_{ij} U(x)| w_j / w_i \right), \end{aligned} \quad (1.4)$$

where  $w = (w_i)_{i=1}^n$  varies over all positive sequences.

Setting  $w_i \equiv 1$  in (1.4), it follows that

$$\lambda_1(U) \geq \inf_{x \in \mathbb{R}^n} \min_{1 \leq i \leq n} \left( \lambda_1^{x_{\setminus i}} - \sum_{j:j \neq i} |\partial_{ij} U(x)| \right) \geq \min_{1 \leq i \leq n} \left[ \inf_{x \in \mathbb{R}^n} \lambda_1^{x_{\setminus i}} - \sum_{j:j \neq i} \|\partial_{ij} U\|_\infty \right].$$

The last lower bound is more or less the estimate given in [5] and [7], goes back to [3].

The supremum over  $w$  in (1.4) comes from a variational formula for the principal eigenvalue of a symmetric  $Q$ -matrix (cf. §3 for more details). The use of the variational formula is necessary, since the principal eigenvalue is not computable in general for a large scale matrix.

The essential point for which (1.4) is valuable is that we now have quite complete knowledge about the spectral gap in dimensional one. For instance, as a consequence of part (1) of Theorem 3.1 in [12], we have

$$\lambda_1^{x \setminus i} \geq \sup_f \inf_{x_i \in \mathbb{R}} \left\{ \partial_{ii} U(x) - \frac{f''(x_i) - \partial_i U(x) f'(x_i)}{f(x_i)} \right\}, \quad (1.5)$$

where  $f$  varies over all positive functions in  $C^2(\mathbb{R})$ . In particular, setting  $f = 1$ , we get

$$\lambda_1^{x \setminus i} \geq \inf_{x_i \in \mathbb{R}} \partial_{ii} U(x). \quad (1.6)$$

When  $\partial_{ii} U(x) = u''(x_i)$  for some  $u \in C^2(\mathbb{R})$ , independent of  $i$ , (1.6) leads to the so-called convex phase condition “ $\inf_{x \in \mathbb{R}} u''(x) > 0$ .” Since a local modification of  $u$  does not change the positiveness of  $\lambda_1$ , the convex condition can be replaced by  $\lim_{|x| \rightarrow \infty} u''(x) > 0$  (i.e., the convexity at infinity) as proved in [12; Corollary 3.5], see also Theorem 4.1 below. However, the last condition is still not necessary as shown by [12; Example 3.11 (3):  $u'(x) = \gamma x(\gamma + \cos x)^{-1}$  for some  $\gamma > 1$ ] and [5; Proposition 4.4] (see also Example 2.5 below). A more careful examination of spectral gap in dimension one is delayed to §4.

It is possible to avoid the use of test functions  $w$  and  $f$  in (1.4) and (1.5), respectively. To see this, we introduce an explicit lower estimate of  $\lambda_1(U)$ . For this, we need additional notations. Choose a practical  $\eta_i^{x \setminus i} \leq \lambda_1^{x \setminus i}$ , as bigger as possible, and define

$$\begin{aligned} s_i(x) &= \eta_i^{x \setminus i} - \sum_{j: j \neq i} |\partial_{ij} U(x)|, & \underline{s}(x) &= \min_{1 \leq i \leq n} s_i(x), \\ q_i(x) &= \eta_i^{x \setminus i} - \underline{s}(x), & d_i(x) &= s_i(x) - \underline{s}(x), \\ h^{(\gamma)}(x) &= \min_{A: \emptyset \neq A \subset \{1, 2, \dots, n\}} \frac{1}{|A|} \left[ \sum_{i \in A} \frac{d_i(x)}{q_i(x)^\gamma} + \sum_{i \in A, j \notin A} \frac{|\partial_{ij} U(x)|}{[q_i(x) \vee q_j(x)]^\gamma} \right], \\ \gamma &\geq 0, \end{aligned} \quad (1.7)$$

where  $a \vee b = \max\{a, b\}$  and  $|A|$  is the cardinality of the set  $A$ .

**Theorem 1.2.** *We have*

$$\lambda_1(U) \geq \inf_{x \in \mathbb{R}^n} \left\{ \underline{s}(x) + \frac{h^{(1/2)}(x)^2}{1 + \sqrt{1 - h^{(1)}(x)^2}} \right\}. \quad (1.8)$$

A close related topic to the Poincaré inequality is the logarithmic Sobolev inequality with optimal constant  $\sigma(U)$ :

$$\sigma(U) \text{Ent}_\mu(f^2) \leq 2D(f), \quad f \in \mathcal{D}(D), \quad (1.9)$$

where  $\text{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f)$  for  $f \geq 0$ . Correspondingly, we have the conditional marginal inequality for  $\mu_U^{x_{\setminus i}}$ , given  $x_{\setminus i}$ , with optimal constant  $\sigma^{x_{\setminus i}}$ :

$$\sigma^{x_{\setminus i}} \text{Ent}_{\mu_U^{x_{\setminus i}}}(f) \leq \int_{\mathbb{R}} f'^2 d\mu_U^{x_{\setminus i}}, \quad f \in C_0^\infty(\mathbb{R}). \quad (1.10)$$

We can now state a very recent result due to [8; Theorem 1], which is consistent with Theorem 1.1.

**Theorem 1.3.** *The logarithmic Sobolev constant  $\sigma(U) \geq \lambda_{\min}(A)$ , where the matrix  $A = (A_{ij})$  is defined by*

$$A_{ij} = \begin{cases} \inf_x \sigma^{x_{\setminus i}}, & \text{if } j = i \\ -\sup_x |\partial_{ij} U(x)|. & \text{if } j \neq i. \end{cases}$$

In view of the above results, it is clear that the one-dimensional case plays a crucial role. In that case, a representative result of the paper is as follows.

**Proposition 1.4.** *In dimensional one, replace  $U$  with  $u_{\beta_1, \beta_2}(x) = x^4 - \beta_1 x^2 + \beta_2 x$  for some constants  $\beta_1 \geq 0$  and  $\beta_2 \in \mathbb{R}$ . Then we have*

$$\begin{aligned} 4e^{14} \exp \left[ -\frac{1}{4} \beta_1^2 + 2 \log(1 + \beta_1) \right] &\geq \inf_{\beta_2} \lambda_1(u_{\beta_1, \beta_2}) \\ &\geq \inf_{\beta_2} \sigma(u_{\beta_1, \beta_2}) \\ &\geq \frac{\sqrt{\beta_1^2 + 8} - \beta_1}{\sqrt{e}} \exp \left[ -\frac{1}{8} \beta_1 (\beta_1 + \sqrt{\beta_1^2 + 8}) \right]. \end{aligned}$$

*In particular,  $\inf_{\beta_2} \lambda_1(u_{\beta_1, \beta_2})$  and  $\inf_{\beta_2} \sigma(u_{\beta_1, \beta_2})$  have the same order as  $\exp[-\beta_1^2/4 + O(\log \beta_1)]$  as  $\beta_1 \rightarrow \infty$ .*

The exponent  $\beta_1^2/4$  here equals, approximately as  $\beta_1 \rightarrow \infty$ , the square of the variance of a random variable having the distribution with density  $\exp[-x^4 + \beta_1 x^2]/Z$  on the real line.

The remainder of the paper is organized as follows. In the next section, we study an alternative variational formula for spectral gap. This is especially meaningful in the context of diffusions. The proofs of Theorems 1.1 and 1.2 are completed in §3. The one-dimensional spectral gap is the main topic in §4. The logarithmic Sobolev constant is studied in §5, in which Proposition 1.4 is proven. Even though the explicit and universal upper and lower estimates, as well as the criteria, for the spectral gap and logarithmic Sobolev constant are all known (cf. [13; Chapter 5, Theorem 7.4] and §4 below), it is still quite a distance to arrive at Proposition 1.4. Actually, we study this model several times (Examples 4.3, 4.6, 4.9, 5.3, and Proposition 4.7) by using different approaches. Thus, a part of the paper is methodological, it takes time and space to make some comparison of different methods. Two typical infinite-dimensional models are treated in the last section.

**2. Alternative variational formula for spectral gap.** Let  $(E, \mathcal{E}, \mu)$  be a probability space and  $L^2(\mu)$  be the ordinary  $L^2$ -space of real functions. Corresponding to a  $\mu$ -reversible Markov process with transition probability  $P(t, x, \cdot)$ , we have a positive, strongly continuous, contractive and self-adjoint semigroup  $\{P_t\}_{t \geq 0}$  on  $L^2(\mu)$  with generator  $(L, \mathcal{D}(L))$ . Throughout this section,  $(\cdot, \cdot)$  and  $\|\cdot\|$  denote, respectively, the inner product and the norm in  $L^2(\mu)$ . By elementary spectral theory, we have

$$\frac{1}{t}(f - P_t f, f) \uparrow \text{ some } D(f, f) =: D(f) \leq \infty \text{ as } t \downarrow 0. \quad (2.1)$$

Set  $\mathcal{D}(D) = \{f \in L^2(\mu) : D(f) < \infty\}$  and define  $D(f, g) = (D(f+g) - D(f-g))/4$  for  $f, g \in \mathcal{D}(D)$ . Then,  $(D, \mathcal{D}(D))$  is a Dirichlet form. Moreover,

$$D(f, g) = -(Lf, g), \quad f, g \in \mathcal{D}(L). \quad (2.2)$$

The formula in (2.4) below goes back to [14].

**Theorem 2.1.** *The spectral gap  $\lambda_1(L)$  is described by the largest constant  $\kappa$  in the following equivalent inequalities.*

$$\kappa \text{Var}_\mu(f) \leq D(f), \quad f \in \mathcal{D}(D), \quad (2.3)$$

$$\kappa D(f) \leq \|Lf\|^2, \quad f \in \mathcal{D}(L). \quad (2.4)$$

*Proof.* Let  $\{E_\alpha\}_{\alpha \geq 0}$  be the spectral representation of  $L$ . Then  $L = -\int_0^\infty \alpha dE_\alpha$ . The optimal constant  $\kappa$  in (2.3) is known to be  $\lambda_1 = \lambda_1(L)$ . Note that

$$\begin{aligned} \|Lf\|^2 &= (Lf, Lf) \\ &= (f, L^2 f) \\ &= \left( f, \int_0^\infty \alpha^2 dE_\alpha f \right) \\ &= \int_0^\infty \alpha^2 d(E_\alpha f, f) \\ &= \int_{\lambda_1}^\infty \alpha^2 d(E_\alpha f, f) \geq \lambda_1 \int_{\lambda_1}^\infty \alpha d(E_\alpha f, f) \\ &= \lambda_1 \int_0^\infty \alpha d(E_\alpha f, f) \\ &= \lambda_1 (f, -Lf) \\ &= \lambda_1 D(f). \end{aligned}$$

Because the only inequality here cannot be improved, the largest constant  $\kappa$  in (2.4) is also equal to  $\lambda_1$ .  $\square$

**Remark 2.2.** Actually, it is known and is also easy to check that (2.3) is equivalent to the correlation inequality

$$\lambda_1(L)|\text{Cov}_\mu(f, g)| \leq (D(f)D(g))^{1/2}, \quad f, g \in \mathcal{D}(D), \quad (2.5)$$

where  $\text{Cov}_\mu(f, g) = \mu(fg) - \mu(f)\mu(g)$  and  $\mu(f) = \int f d\mu$ . See the comment below Proposition 3.2 for a proof.

Before moving further, let us mention that the above proof also works for the principal eigenvalue. In this case,  $L1 \neq 0$  and  $\mu$  can be infinite. Then the principal eigenvalue  $\lambda_0$  can be described by the following equivalent inequalities.

$$\begin{aligned} \tilde{\kappa}\|f\|^2 &\leq D(f), & f \in \mathcal{D}(D), \\ \tilde{\kappa}D(f) &\leq \|Lf\|^2, & f \in \mathcal{D}(L). \end{aligned} \quad (2.6)$$

The formula (2.4) is especially useful for diffusion on Riemannian manifolds. Thus, the next result is meaningful for a more general class of diffusion in  $\mathbb{R}^n$  by using a suitable Riemannian structure.

**Corollary 2.3.** Let  $L = \Delta - \langle \nabla U, \nabla \rangle$  for some  $U \in C^\infty(\mathbb{R}^n)$  with  $Z := \int_{\mathbb{R}^n} e^{-U} dx < \infty$  and set  $\mu(dx) = e^{-U} dx / Z$ . Then

$$\|Lf\|^2 = \int_{\mathbb{R}^n} \left[ \sum_{i,j} (\partial_{ij} f)^2 + \langle \text{Hess}(U) \nabla f, \nabla f \rangle \right] d\mu, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (2.7)$$

where  $\langle \cdot, \cdot \rangle$  stands the usual inner product in  $\mathbb{R}^n$ . In particular, we have

$$\lambda_1(U) \geq \inf_{x \in \mathbb{R}^n} \lambda_{\min}(\text{Hess}(U)(x)), \quad (2.8)$$

where  $\lambda_{\min}(M)$  is the minimal eigenvalue of the matrix  $M$ .

*Proof.* The proof of (2.7) is mainly a use of integration by parts formula. Because  $Lf = \sum_i (\partial_{ii} f - \partial_i U \partial_i f)$ , we have

$$\langle \nabla f, \nabla Lf \rangle = \sum_j \partial_j f \sum_i \partial_j (\partial_{ii} f - \partial_i U \partial_i f) = \sum_{i,j} \partial_j f (\partial_{iij} f - \partial_{ij} f \partial_i U - \partial_i f \partial_{ij} U).$$

Next,

$$\begin{aligned} \frac{1}{Z} \int_{\mathbb{R}^n} \sum_j \partial_j f \sum_i (\partial_{iij} f - \partial_{ij} f \partial_i U) e^{-U} &= \frac{1}{Z} \int_{\mathbb{R}^n} \sum_j \partial_j f \sum_i \partial_i (\partial_{ij} f e^{-U}) \\ &= -\frac{1}{Z} \int_{\mathbb{R}^n} \sum_{i,j} (\partial_{ij} f)^2 e^{-U} \\ &= - \int_{\mathbb{R}^n} \sum_{i,j} (\partial_{ij} f)^2 d\mu, \quad f \in C_0^\infty(\mathbb{R}^n). \end{aligned}$$

Noting that  $\mu$  is a probability measure and the diffusion coefficients are constants, the Dirichlet form is regular (cf. [12; condition (4.13)] for instance). Actually, the martingale problem for  $L$  is well posed. Thus,  $LC_0^\infty(\mathbb{R}^n) \subset C_0^\infty(\mathbb{R}^n) \subset \mathcal{D}(L)$ , and so

$$\|Lf\|^2 = \int_{\mathbb{R}^n} Lf \cdot Lf d\mu = - \int_{\mathbb{R}^n} \langle \nabla Lf, \nabla f \rangle d\mu, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Combining these facts together, we get (2.7).

To prove the last assertion, applying Theorem 2.1 and (2.7), we get

$$\begin{aligned} \lambda_1(L) &= \inf_{f \in \mathcal{D}(L), f \neq \text{const}} \frac{\|Lf\|^2}{D(f)} \\ &= \inf_{f \in C_0^\infty(\mathbb{R}^n), f \neq \text{const}} \frac{\|Lf\|^2}{D(f)} \\ &\geq \inf_{f \in C_0^\infty(\mathbb{R}^n), f \neq \text{const}} \int_{\mathbb{R}^n} \langle \text{Hess}(U) \nabla f, \nabla f \rangle d\mu / D(f) \\ &\geq \inf_{x \in \mathbb{R}^n} \lambda_{\min}(\text{Hess}(U)(x)). \quad \square \end{aligned}$$

**Remark 2.4.** Actually, under the assumption of Corollary 2.3, the Bakry-Emery criterion (cf. [14] or [7; Corollary 1.6]) implies a stronger conclusion:

$$\sigma(U) \geq \inf_{x \in \mathbb{R}^n} \lambda_{\min}(\text{Hess}(U)(x)). \quad (2.9)$$

A simple counterexample for which (2.8) and (2.9) are not effective is the following. This example also shows that (1.4) is an improvement of (2.8).

**Example 2.5.** Consider the two-dimensional case. Let

$$U(x) = x_1^4 + x_2^4 - \beta(x_1^2 + x_2^2) + 2Jx_1x_2$$

with constants  $\beta \geq 0$  and  $J \in \mathbb{R}$ . Then  $\inf_{x \in \mathbb{R}^2} \lambda_{\min}(\text{Hess}(U)(x)) \leq 0$  and  $U$  is not convex at infinity, but  $\lambda_1(U) > 0$  in a region of  $(\beta, J) \subset \mathbb{R} \times \mathbb{R}_+$ .

*Proof.* First, we have

$$\text{Hess}(U)(x) = \begin{pmatrix} 12x_1^2 - 2\beta & 2J \\ 2J & 12x_2^2 - 2\beta \end{pmatrix}.$$

Because for the matrix

$$A = \begin{pmatrix} c_1 & 2J \\ 2J & c_2 \end{pmatrix},$$

we have  $\lambda_{\min}(A) = 2^{-1}(c_1 + c_2 - \sqrt{(c_1 - c_2)^2 + 16J^2})$ . Hence

$$\lambda_{\min}(\text{Hess}(U)(x)) = 2 \min_{x_1, x_2} \left\{ 3(x_1^2 + x_2^2) - \beta - \sqrt{9(x_1^2 - x_2^2)^2 + J^2} \right\}.$$

Setting  $x_1 = x_2 = 0$ , we get

$$\inf_{x \in \mathbb{R}^2} \lambda_{\min}(\text{Hess}(U)(x)) \leq -2(\beta + |J|) \leq 0.$$

Next, since

$$\lim_{|x_1| \rightarrow \infty} \left( 3x_1^2 - \beta - \sqrt{9x_1^4 + J^2} \right) = \lim_{z \rightarrow 0} \frac{3 - \sqrt{9 + J^2 z^2}}{z} - \beta = -\beta,$$

we have

$$\lim_{|x| \rightarrow \infty} \lambda_{\min}(\text{Hess}(U)(x)) \leq \lim_{x_2=0, |x_1| \rightarrow \infty} \lambda_{\min}(\text{Hess}(U)(x)) \leq -2\beta \leq 0.$$

This means that  $U$  is not convex at infinity. The last assertion of the example is the one of the main aims of this paper and it is even true in the higher dimensions (cf. Theorem 6.3 below).  $\square$

**3. Proofs of Theorems 1.1 and 1.2 and some remarks.** As a preparation, we prove a result which is an improvement of (2.8) and [7; Proposition 3.1]. We adopt the notation given in §1.

**Proposition 3.1.** *We have*

$$\lambda_1(U) \geq \inf_{x \in \mathbb{R}^n} \lambda_{\min}(\widetilde{\text{Hess}}(U)(x)). \quad (3.1)$$

*Proof.* First, applying Theorem 2.1 and (2.7) to the  $i$ th marginal, we have

$$\int_{\mathbb{R}} [(\partial_{ii}f)^2 + (\partial_{ii}U)(\partial_i f)^2] d\mu_U^{x_{\setminus i}} \geq \lambda_1^{x_{\setminus i}} \int_{\mathbb{R}} (\partial_i f)^2 d\mu_U^{x_{\setminus i}}, \quad f \in C_0^\infty(\mathbb{R}^n). \quad (3.2)$$

Next, denote by  $\text{Hess}_0(U)$  the symmetric matrix obtained from the Hessian matrix  $\text{Hess}(U)$  replacing the diagonal elements with zero. Then, by (3.2), we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left[ \sum_{i,j} (\partial_{ij}f)^2 + \langle \text{Hess}(U) \nabla f, \nabla f \rangle \right] d\mu_U \\ & \geq \sum_i \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}} [(\partial_{ii}f)^2 + (\partial_{ii}U)(\partial_i f)^2] d\mu_U^{x_{\setminus i}} \right\} d\mu_U - \sum_i \int_{\mathbb{R}^n} [(\partial_{ii}U)(\partial_i f)^2] d\mu_U \\ & \quad + \int_{\mathbb{R}^n} \langle \text{Hess}(U) \nabla f, \nabla f \rangle d\mu_U \\ & \geq \sum_i \int_{\mathbb{R}^n} \left\{ \lambda_1^{x_{\setminus i}} \int_{\mathbb{R}} (\partial_i f)^2 d\mu_U^{x_{\setminus i}} \right\} d\mu_U + \int_{\mathbb{R}^n} \langle \text{Hess}_0(U) \nabla f, \nabla f \rangle d\mu_U \\ & = \sum_i \int_{\mathbb{R}^n} \lambda_1^{x_{\setminus i}} (\partial_i f)^2 d\mu_U + \int_{\mathbb{R}^n} \langle \text{Hess}_0(U) \nabla f, \nabla f \rangle d\mu_U \\ & = \int_{\mathbb{R}^n} \langle \widetilde{\text{Hess}}(U) \nabla f, \nabla f \rangle d\mu_U \\ & \geq \inf_{x \in \mathbb{R}^n} \lambda_{\min}(\widetilde{\text{Hess}}(U)(x)) \int_{\mathbb{R}^n} |\nabla f|^2 d\mu_U, \quad f \in C_0^\infty(\mathbb{R}^n). \end{aligned} \quad (3.3)$$



Now, the required assertion follows from the proof of the last assertion of Corollary 2.3.  $\square$

From the proof of Proposition 3.1, it is clear that the only argument where we may lose somewhat is the first inequality of (3.3), since the terms  $\sum_{i \neq j} (\partial_{ij} f)^2$  are ignored there. Hence the estimate (3.1) is mainly meaningful if the interactions are not strong. The interacting potentials considered in this paper are rather simple; for general interactions, one needs some “block estimates” which are not touched here, instead of the “single-site estimates” studied in this paper.

The shorthand of (3.1) is that the minimal eigenvalue  $\lambda_{\min}(\widetilde{\text{Hess}}(U))$  may not be computable in practice. For this, we need the second variational procedure. To do so, let  $s = \min_i \{\lambda_1^{x \setminus i} - \sum_{j: j \neq i} |\partial_{ij} U|\}$  and define

$$q_{ij} = \begin{cases} |\partial_{ij} U|, & \text{if } i \neq j \\ s - \lambda_1^{x \setminus i}, & \text{if } i = j. \end{cases}$$

Then,  $Q := (q_{ij})$ , depending on  $x$ , is a symmetric  $Q$ -matrix, not necessarily conservative (i.e.,  $\sum_j q_{ij} \leq 0$ ).

*Proof of Theorem 1.1.* The first estimate in (1.4) follows from Proposition 3.1. Next, by [15; Theorem 1.1], we have

$$\lambda_{\min}(-Q) \geq \sup_{w > 0} \min_i [-Qw/w](i), \quad (3.4)$$

where  $Qw(i) = \sum_j q_{ij} w_j$ . We remark that the sign of the equality in (3.4) holds once  $Q$  is irreducible (cf. [15; Proposition 4.1]). Noting that for every symmetric matrix  $B = (b_{ij})$  with nonnegative diagonals and any vector  $w$ , we have

$$\langle w, Bw \rangle = \sum_i b_{ii} w_i^2 + 2 \sum_{i \neq j} b_{ij} w_i w_j \geq \sum_i b_{ii} w_i^2 - 2 \sum_{i \neq j} |b_{ij} w_i w_j| = \langle |w|, \tilde{B}|w| \rangle,$$

where  $\tilde{B} = (\tilde{b}_{ij}) : \tilde{b}_{ii} = b_{ii}$ ,  $\tilde{b}_{ij} = -|b_{ij}|$  for  $i \neq j$  and  $|w| = (|w_i|)$ . Letting  $w^*$  be a vector with  $\langle w^*, w^* \rangle = 1$  such that  $\lambda_{\min}(B) = \langle w^*, Bw^* \rangle$ , it follows that

$$\lambda_{\min}(B) \geq \langle |w^*|, \tilde{B}|w^*| \rangle \geq \lambda_{\min}(\tilde{B}) \langle |w|^*, |w|^* \rangle = \lambda_{\min}(\tilde{B}).$$

Based on this fact and as an application of (3.4), we get

$$\begin{aligned} \lambda_{\min}(\widetilde{\text{Hess}}(U)(x)) &\geq \lambda_{\min}(\text{diag}(s) - Q) \\ &= s + \lambda_{\min}(-Q) \\ &\geq s + \max_{w > 0} \min_i \left[ -s + \lambda_1^{x \setminus i} - \sum_{j: j \neq i} q_{ij} w_j / w_i \right] \\ &= \max_{w > 0} \min_i \left[ \lambda_1^{x \setminus i} - \sum_{j: j \neq i} q_{ij} w_j / w_i \right]. \end{aligned} \quad (3.5)$$

Combining this with the first estimate in (1.4), we get the second one in (1.4), and so complete the proof of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* In the above proof, replacing  $\lambda_1^{x \setminus i}$ ,  $s$  and  $q_{ii}$  with  $\eta_i^{x \setminus i}$ ,  $\underline{s}(x)$  and  $q_{ii}(x)$ , respectively, but keep  $q_{ij}$  ( $i \neq j$ ) to be the same, we obtain

$$\lambda_{\min}(\widetilde{\text{Hess}}(U)(x)) \geq \underline{s}(x) + \lambda_{\min}(-Q(x)).$$

By Proposition 3.1, it suffices to estimate  $\lambda_{\min}(-Q(x))$ . Note that  $\lambda_{\min}(-Q(x))$  is nothing but the principal (Dirichlet) eigenvalue of  $Q(x)$ , often denoted by  $\lambda_0(Q(x))$ . Because  $Q(x)$  is symmetric, and so its symmetrizing measure is just the uniform distribution on  $\{1, 2, \dots, n\}$ . Now the conclusion of Theorem 1.2 follows from [16; Theorem 1.1] plus some computations.  $\square$

We conclude this section with some remarks.

Let  $d_i = -q_{ii} - \sum_{j \neq i} q_{ij}$ . By setting  $w_i = \text{constant}$  in (3.4), it follows that  $\lambda_{\min}(-Q) \geq \min_i d_i$ . The sign of the equality holds if  $(d_i)$  is a constant. Otherwise, this well-known simplest conclusion is usually rough. For instance, take

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}.$$

Then  $\lambda_{\min}(-Q) = 2 - \sqrt{3} > 0$  (the equality of (3.4) is attained at the positive eigenvector  $w = (2 + \sqrt{3}, 1 + \sqrt{3}, 1)$ ) but  $\min_i d_i = 0$ . This shows that the use of the variational formula (3.4) is necessary to produce sharper lower bounds.

When  $\partial_{ij}U \leq 0$  for all  $i \neq j$ , then  $\widetilde{\text{Hess}}(U) = \text{diag}(s) - Q$ , and so the sign of the first equality in (3.5) holds. In this case, the estimate (3.5) is quite sharp, since so is (3.4). However, for general  $\partial_{ij}U$  ( $i \neq j$ ), the lower bound in (3.5) may be less effective but we do not have a variational formula as (3.4) in such a general situation.

For a given symmetric matrix  $B = (b_{ij})$  ( $\widetilde{\text{Hess}}(U)$ , for instance), the classical variational formula, which is especially powerful for upper bounds, is as follows.

$$\begin{aligned} \lambda_{\min}(B) &= \inf \left\{ \sum_{i,j} w_i b_{ij} w_j : \sum_i w_i^2 = 1 \right\} \\ &= \inf \left\{ \sum_i \left( b_{ii} + \sum_{j:j \neq i} b_{ij} \right) w_i^2 - \frac{1}{2} \sum_{i,j} b_{ij} (w_j - w_i)^2 : \sum_i w_i^2 = 1 \right\}. \end{aligned} \tag{3.6}$$

For a given symmetrizable  $Q$ -matrix  $(q_{ij})$  with symmetric probability measure  $\mu$ , set

$$D(f) = \frac{1}{2} \sum_{i,j} \mu_i q_{ij} (f_j - f_i)^2 + \sum_i \mu_i d_i f_i^2,$$

where  $d_i = -q_{ii} - \sum_{j \neq i} q_{ij}$  as defined before. Then, an alternative formula of (3.6), in terms of the Donsker-Varadhan's theory of large deviations, goes as follows.

$$\begin{aligned}
\lambda_{\min}(-Q) &= \inf_f \left\{ D(f) : \sum_i \mu_i f_i^2 = 1 \right\} \\
&= \inf_{\alpha \geq 0} \left\{ D(\sqrt{d\alpha/d\mu}) : \sum_i \alpha_i = 1 \right\} \\
&= \inf_{\alpha \geq 0} \left\{ I(\alpha) + \sum_i \alpha_i d_i : \sum_i \alpha_i = 1 \right\} \\
&= \inf_{\alpha \geq 0} \left\{ - \inf_{u > 0} \sum_{i,j} \alpha_i q_{ij} (u_j - u_i)/u_i + \sum_i \alpha_i d_i : \sum_i \alpha_i = 1 \right\} \\
&= \inf_{\alpha \geq 0} \left\{ \frac{1}{2} \sum_{i,j} (\sqrt{\alpha_i q_{ij}} - \sqrt{\alpha_j q_{ji}})^2 + \sum_i \alpha_i d_i : \sum_i \alpha_i = 1 \right\},
\end{aligned} \tag{3.7}$$

where  $I$  is the  $I$ -functional in the theory of large deviations. Refer to [17; Proof of Theorem 8.17] for more details. In other words, the large deviation principle provides an alternative description of the classical variational formula, but not (3.4), for which one needs a variational formula for the Dirichlet forms (cf. [15]).

Finally, we remark that the proof of Proposition 3.1 can be also used in the study of other inequalities. The details are omitted here since they are not used subsequently (cf. [7]). The next one is a partial extension of (2.5).

**Proposition 3.2.** *Under the assumption of Corollary 2.3, we have for every invertible, nonnegative and diagonal matrix  $D$ , the largest constant  $\kappa$ :*

$$\kappa |\text{Cov}_{\mu_U}(f, g)| \leq \left( \int |D\nabla f|^2 d\mu_U \int |D^{-1}\nabla f|^2 d\mu_U \right)^{1/2}, \tag{3.8}$$

$f, g \in C_0^\infty(\mathbb{R}^n)$

satisfies

$$\kappa \geq \inf_x \hat{\lambda}_{\min}(D \widetilde{\text{Hess}}(U) D^{-1}(x)), \tag{3.9}$$

where  $\hat{\lambda}_{\min}(M) = \max\{c : M \geq c \text{Id}\}$ .

Before moving further, let us make some remarks about the proof of Proposition 3.2. Note that

$$\int |D\nabla f|^2 d\mu = \int \langle D^2 \nabla f, \nabla f \rangle d\mu$$

which is the Dirichlet form corresponding to the diffusion operator with diffusion coefficients  $D^2$  and potential  $U$ . Denote by  $\lambda_1(D^2, U)$  the spectral gap of the last operator, then we have

$$\begin{aligned}
|\text{Cov}_\mu(f, g)|^2 &\leq \text{Var}_\mu(f) \text{Var}_\mu(g) \\
&\leq \frac{1}{\lambda_1(D^2, U) \lambda_1(D^{-2}, U)} \int \langle D^2 \nabla f, \nabla f \rangle d\mu \int \langle D^{-2} \nabla g, \nabla g \rangle d\mu.
\end{aligned} \tag{3.10}$$

Hence, we obtain a lower bound of the optimal constant in (3.8):

$$\kappa \geq \sqrt{\lambda_1(D^2, U) \lambda_1(D^{-2}, U)}. \quad (3.11)$$

The proof is quite natural. Furthermore, by setting  $D$  to be the identity matrix, we obtain (2.5) with sharp constant. However, the estimate (3.11) is usually not sharp in the general case. Note that the sign of the last equality in (3.11) holds if  $f$  and  $g$  are the correspondent eigenfunctions with respect to the operators, but the sign of the first equality in (3.11) holds iff  $f$  and  $g$  are proportional almost surely (due to the use of the Cauchy-Schwarz inequality). This can happen only if  $D$  is trivial: all the diagonals of  $D$  are equal.

A better way to study (3.8) is using the semigroup's approach. Write

$$\begin{aligned} \text{Cov}_\mu(f, g) &= \int (f - \mu(f))g d\mu \\ &= - \int \left( \int_0^\infty \frac{d}{dt} P_t f dt \right) g d\mu \\ &= - \int_0^\infty \left( \int g L P_t f d\mu \right) dt \\ &= \int_0^\infty \left( \int \langle \nabla P_t f, \nabla g \rangle d\mu \right) dt. \end{aligned}$$

Now, as a good application of the Cauchy-Schwarz inequality, we get

$$|\text{Cov}_\mu(f, g)| \leq \left[ \int_0^\infty \left( \int |D \nabla P_t f|^2 d\mu \right)^{1/2} dt \right] \left( \int |D^{-1} \nabla g|^2 d\mu \right)^{1/2}.$$

The problem is now reduced to study the decay of  $\int |D \nabla P_t f|^2 d\mu$  in  $t$  (cf. [7]).

Similarly to Proposition 3.1, as checked by Feng Wang in 2002, we have the following result which improves (2.9), but may be weaker than Theorem 1.3.

**Proposition 3.3.** *Under the assumption of Corollary 2.3, we have*

$$\sigma(U) \geq \inf_x \lambda_{\min}(\overline{\text{Hess}}(U)), \quad (3.12)$$

where

$$(\overline{\text{Hess}}(U))_{ij} = \begin{cases} \zeta^{x_{\setminus i}}, & \text{if } j = i \\ (\text{Hess}(U))_{ij} & \text{if } j \neq i, \end{cases}$$

$\zeta^{x_{\setminus i}}$  is the optimal constant in the inequality

$$\zeta^{x_{\setminus i}} \int f(\partial_i \log f)^2 d\mu_U^{x_{\setminus i}} \leq \int f \Gamma^i(\log f) d\mu_U^{x_{\setminus i}}, \quad 0 \leq f \in C_0^\infty(\mathbb{R}^n), \quad (3.13)$$

and

$$\Gamma^i(f) = (\partial_{ii} f)^2 + (\partial_{ii} U)(\partial_i f)^2.$$

**4. One-dimensional case. Explicit estimates.** The operator now becomes  $L = d^2/dx^2 - u'(x)d/dx$ . Write  $b(x) = -u'(x)$ . Then  $b$  must have a real root. Otherwise, without loss of generality, let  $u' \geq \varepsilon > 0$ . Then  $-u$  is strictly decreasing, and so

$$\infty > Z := \int_{\mathbb{R}} e^{-u} \geq \int_{-\infty}^0 e^{-u} > e^{-u(0)} \int_{-\infty}^0 1 = \infty,$$

which is a contradiction.

Unless otherwise stated, throughout this section, we consider the operator

$$L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}.$$

Assume that  $a \in C(\mathbb{R})$ ,  $a > 0$  and  $Z = \int_{\mathbb{R}} e^{C(x)}/a(x) < \infty$ , where  $C(x) = \int_0^x b/a$ . Define  $\mu(dx) = (Za(x))^{-1} e^{C(x)} dx$ . Recall that

$$\lambda_1(L) = \inf\{D(f) : f \in C^1(\mathbb{R}), \mu(f) = 0, \mu(f^2) = 1\},$$

where  $D(f) = \int_{\mathbb{R}} a f'^2 d\mu$ .

Let  $\theta$  be a fixed real root of  $b$ . Choose  $K = K_{\theta} \in C(\mathbb{R} \setminus \{\theta\})$  such that  $K$  is increasing (i.e., non-decreasing) in  $x$  when  $|x - \theta|$  increases,  $K(\theta \pm 0) > -\infty$ , and moreover

$$K(r) \leq \inf_{x: \pm(x-r) > 0} [-b(x)/(x - \theta)] \quad \text{for all } \pm(r - \theta) > 0, \quad (4.1)$$

where and in what follows, the notation “ $\pm$ ” means that there are two cases: one takes “ $+$ ” (resp., “ $-$ ”) everywhere in the statement. Define

$$F(s) = F^r(s) = \int_{\theta}^s \frac{u - \theta}{a(u)} [K(r) - K(u)] du, \quad s, r \in \mathbb{R}, \quad (4.2)$$

$$\delta_{\pm}(K) = \sup_{r: \pm(r-\theta) > 0} K(r) \inf_{s: \pm(r-\theta) > \pm(s-\theta) > 0} \frac{(s - \theta) \exp[-F(s)]}{\int_{\theta}^s \exp[-F(u)] du} \quad (4.3)$$

$$\geq \sup_{r: \pm(r-\theta) > 0} K(r) \exp[-F(r)]. \quad (4.4)$$

The next result is a modification of [12; Corollary 3.5]. It is specially useful for those  $b$  growing at least linear.

**Theorem 4.1.**

(1) *By using the above notations, we have*

$$\lambda_1(L) \geq \delta_+(K) \wedge \delta_-(K). \quad (4.5)$$

(2) *Suppose additionally that  $K$  is a piecewise  $C^1$ -function, then we have*

$$\delta_{\pm}(K) \geq K(r_{\pm}) \exp \left[ - \int_{\theta}^{r_{\pm}} \left( \int_{\theta}^x \frac{u - \theta}{a(u)} du \right) dK(x) \right], \quad (4.6)$$

where

$$r_{\pm} = \pm\infty, \quad \text{if } \lim_{r \rightarrow \pm\infty} K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du \leq 1, \quad (4.7)$$

and otherwise,  $r_{\pm}$  is the unique solution to the equation

$$K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du = 1, \quad \pm(r - \theta) > 0. \quad (4.8)$$

*Proof.* (a) First, consider the half-line  $(\theta, \infty)$ . Assume that  $K(r_1) > 0$  for some  $r_1 \in (\theta, \infty)$ . Otherwise, (4.5) becomes trivial. Fix  $r = r_1$  and define

$$f_+(x) = \int_{\theta}^x dy \exp[-F(y \wedge r_1)], \quad x \geq \theta.$$

Then, we have  $f_+ > 0$  on  $(\theta, \infty)$ ,  $f_+(\theta) = 0$ ,  $f'_+(\theta) = 1$  and

$$\begin{aligned} f'_+(x) &= \exp[-F(x \wedge r_1)] > 0, \\ f''_+(x) &= -\frac{x - \theta}{a(x)} [K(r_1) - K(x \wedge r_1)] f'_+(x) \leq 0, \quad x \geq \theta. \end{aligned}$$

Since  $a \in C(\mathbb{R})$ ,  $a > 0$ ,  $K \in C(\mathbb{R} \setminus \{\theta\})$  and  $K(\theta+)$  is finite, we have  $f_+ \in C^2(\theta, \infty)$ .

Next, because  $K$  is increasing on  $(\theta, \infty)$  and  $K(x) \leq -b(x)/(x - \theta)$  for all  $x > \theta$ , we have

$$\begin{aligned} -(af''_+ + bf'_+)(x) &= \{(x - \theta)[K(r_1) - K(x \wedge r_1)] - b(x)\} f'_+(x) \\ &\geq \{(x - \theta)K(r_1) - (x - \theta)K(x) - b(x)\} f'_+(x) \\ &\geq (x - \theta)K(r_1) f'_+(x), \quad x > \theta. \end{aligned} \quad (4.9)$$

Since  $f''_+ \leq 0$ ,  $f'_+$  is decreasing. By the Cauchy mean value theorem, it follows that  $(x - \theta)/f_+(x)$  is increasing on  $(\theta, \infty)$ . Hence, by (4.9), we obtain

$$\begin{aligned} -\left[\frac{af''_+ + bf'_+}{f_+}\right](x) &\geq \frac{r_1 - \theta}{f_+(r_1)} K(r_1) f'_+(x) = \frac{r_1 - \theta}{f_+(r_1)} K(r_1) f'_+(r_1), \\ &\quad x \geq r_1. \end{aligned} \quad (4.10)$$

Combining (4.9) with (4.10), it follows that

$$\inf_{x > \theta} \left[ -\frac{af''_+ + bf'_+}{f_+} \right] \geq K(r_1) \inf_{s \in (\theta, r_1)} \frac{(s - \theta)f'_+(s)}{f_+(s)}.$$

By (4.3), we have thus obtained

$$\inf_{x > \theta} \left[ -\frac{af''_+ + bf'_+}{f_+} \right] \geq \delta_+(K). \quad (4.11)$$

(b) Next, consider the half-line  $(-\infty, \theta)$ . The proof is parallel to (a). Let  $K(r_1) > 0$  for some  $r_1 < \theta$ . Fix  $r = r_1$  and define

$$f_-(x) = \int_{\theta}^x dy \exp[-F(y \vee r_1)], \quad x \leq \theta.$$

Then  $f_- < 0$  on  $(-\infty, \theta)$ ,  $f_-(\theta) = 0$ ,  $f'_- > 0$ ,  $f'_-(\theta) = 1$  and

$$f''_-(x) = -\frac{x-\theta}{a(x)}[K(r_1) - K(x \vee r_1)]f'_-(x) \geq 0$$

for all  $x \leq \theta$ . Moreover  $f_- \in C^2(-\infty, \theta)$ . Then

$$\begin{aligned} -(af''_- + bf'_-)(x) &= \{(x-\theta)[K(r_1) - K(x \vee r_1)] - b(x)\}f'_-(x) \\ &\leq \{(x-\theta)K(r_1) - (x-\theta)K(x) - b(x)\}f'_+(x) \\ &\leq (x-\theta)K(r_1)f'_-(x), \quad x < \theta. \end{aligned}$$

Since  $f_- < 0$  and  $f''_- \geq 0$ , we have

$$-\left[\frac{af''_- + bf'_-}{f_-}\right](x) \geq \frac{r_1 - \theta}{f_-(r_1)}K(r_1)f'_-(x) = \frac{r_1 - \theta}{f_-(r_1)}K(r_1)f'_-(r_1), \quad x \leq r_1.$$

Combining the last two inequalities with (4.3), we get

$$\inf_{x < \theta} \left[ -\frac{af''_- + bf'_-}{f_-} \right] \geq \sup_{r_1 < \theta} K(r_1) \inf_{s \in (\theta, r_1)} \frac{(s-\theta)f'_-(s)}{f_-(s)} = \delta_-(K).$$

Finally, let  $f = f_+I_{[\theta, \infty)} + f_-I_{(-\infty, \theta)}$ . Then  $f \in C^2(\mathbb{R})$  and

$$\begin{aligned} \inf_{x \neq \theta} \left[ -\frac{af'' + bf'}{f} \right](x) &= \left[ \inf_{x > \theta} -\frac{af''_+ + bf'_+}{f_+}(x) \right] \wedge \left[ \inf_{x < \theta} -\frac{af''_- + bf'_-}{f_-}(x) \right] \\ &\geq \delta_+(K) \wedge \delta_-(K). \end{aligned}$$

The estimate (4.5) now follows from the last assertion of [12; Theorem 3.1].

(c) To prove (4.4), noticing that  $K$  is monotone, we may apply the integration by parts formula and rewrite  $F$  as follows.

$$\begin{aligned} F(r) &= \int_{\theta}^r \frac{u-\theta}{a(u)} [K(r) - K(u)] du \\ &= K(r) \int_{\theta}^r \frac{u-\theta}{a(u)} du - \int_{\theta}^r K(u) d\left( \int_{\theta}^u \frac{z-\theta}{a(z)} dz \right) \\ &= \int_{\theta}^r K'(u) \left( \int_{\theta}^u \frac{z-\theta}{a(z)} dz \right) du, \quad r \neq \theta. \end{aligned} \tag{4.12}$$

By the assumption on  $K$ , it follows that  $F \geq 0$ ,  $F(r)$  is increasing in  $r$  as  $|r - \theta|$  increases. Hence, by (4.2), we have

$$\begin{aligned} \delta_{\pm}(K) &\geq \sup_{r: \pm(r-\theta) > 0} K(r) \inf_{s: \pm(r-\theta) > \pm(s-\theta) > 0} \exp[-F(s)] \\ &= \sup_{r: \pm(r-\theta) > 0} K(r) \exp[-F(r)]. \end{aligned}$$

The proof of (4.4) is done.

(d) The second part of the theorem is to compute  $\sup_{r \neq \theta} G(r)$ , where  $G(r) = K(r) \exp[-F(r)]$ . The answer is given by (4.6). To do so, first consider the half line  $(\theta, \infty)$ . Because  $K$  is a piecewisely  $C^1$ , we may assume that  $(\theta, \infty) = \cup_i (c_i, d_i]$ ,  $K \in C^1(c_i, d_i)$  and  $K' \geq 0$  on  $(c_i, d_i)$  for every  $i$ . By (4.12), we have for every  $i$ ,

$$G'(r) = K'(r) \left[ 1 - K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du \right] \exp[-F(r)], \quad r \in (c_i, d_i). \quad (4.13)$$

Let  $\lim_{r \rightarrow \infty} K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du \leq 1$  and set  $\tilde{\theta}_+ = \inf \{r > \theta : K(r) > 0\}$ . Note that  $K$  is increasing,  $K > 0$  on  $(\tilde{\theta}_+, \infty)$ , and so  $K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du$  is strictly increasing on  $(\tilde{\theta}_+, \infty)$ , but is less or equal to zero on  $(\theta, \tilde{\theta}_+)$  when  $\theta < \tilde{\theta}_+$ . It follows that  $K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du \leq 1$  for all  $r \in (\theta, \infty)$ . By (4.13), we have  $G'(r) \geq 0$  on every  $(c_i, d_i)$  since so does  $K'(r)$ . This fact plus the continuity of  $G$  implies that  $\sup_{r > \theta} G(r) = \lim_{r \rightarrow \infty} G(r)$ .

Otherwise, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} K(r) \int_{\theta}^r \frac{u - \theta}{a(u)} du &> 1 \quad \text{and} \\ \lim_{r \rightarrow \tilde{\theta}_+ +} K(r) &\begin{cases} = 0 < \left( \int_{\theta}^{\tilde{\theta}_+} \frac{u - \theta}{a(u)} du \right)^{-1} & \text{if } \theta < \tilde{\theta}_+ \\ < \infty = \left( \int_{\theta}^{\tilde{\theta}_+} \frac{u - \theta}{a(u)} du \right)^{-1} & \text{if } \tilde{\theta}_+ = \theta. \end{cases} \end{aligned}$$

Since  $K(r)$  is increasing and  $\left( \int_{\theta}^r \frac{u - \theta}{a(u)} du \right)^{-1}$  is strictly decreasing, the curves  $K(r)$  and  $\left( \int_{\theta}^r \frac{u - \theta}{a(u)} du \right)^{-1}$  must have uniquely an intersection on  $(\tilde{\theta}_+, \infty)$ , or equivalently on  $(\theta, \infty)$ . So we have  $\sup_{r > \theta} G(r) = \sup_{r > \tilde{\theta}_+} G(r) = G(r_+)$ , where  $r_+$  is the unique solution to the equation (4.8).

The proof of the assertions on  $(-\infty, \theta)$  is parallel.  $\square$

The next two examples illustrate the applications of Theorem 4.1, and are treated several times in the paper.

**Example 4.2.** Let  $u(x) = \alpha x^2 + \beta x$  for some constants  $\alpha > 0$  and  $\beta \in \mathbb{R}$ , and  $a(x) \equiv 1$ . Then we have  $\lambda_1(L_{\alpha, \beta}) \geq \delta_+(K) \wedge \delta_-(K) = 2\alpha$  which is exact.

*Proof.* Since  $-b(x) = -2\alpha x - \beta$ , we have root  $\theta = -\beta/(2\alpha)$ , and so  $-b(x)/(x - \theta) = 2\alpha$ . Thus,  $K(r) = \text{constant } 2\alpha$ . By (4.3), we get  $\delta_{\pm}(K) = 2\alpha$  as claimed. It is easy to check that the estimate is exact, since the corresponding eigenfunction is linear.  $\square$

**Example 4.3.** Let  $u(x) = x^4 - \beta_1 x^2 + \beta_2 x$  for some constants  $\beta_1, \beta_2 \in \mathbb{R}$  and  $a(x) \equiv 1$ . Then we have

$$\lambda_1(L_{\beta_1, \beta_2}) \geq \delta_+(K) \wedge \delta_-(K) \geq \frac{\sqrt{\beta_1^2 + 2} - \beta_1}{\sqrt{e}} \exp \left[ -\frac{1}{2} \beta_1 \left( \beta_1 + \sqrt{\beta_1^2 + 2} \right) \right]$$



uniformly in  $\beta_2$ . When  $\beta_2 = 0$ , we have

$$\lambda_1(L_{\beta_1, \beta_2}) \geq \delta_+(K) \wedge \delta_-(K) \geq \frac{\sqrt{\beta_1^2 + 8} - \beta_1}{\sqrt{e}} \exp \left[ -\frac{1}{8} \beta_1 (\beta_1 + \sqrt{\beta_1^2 + 8}) \right].$$

*Proof.* First, we have  $b(x) = -u'(x) = -4x^3 + 2\beta_1 x - \beta_2$ . Let  $\theta$  be a real root of  $u'$ . For instance, we may take

$$\theta = \begin{cases} 0 & \text{if } \beta_2 = 0 \\ -\left(\frac{\beta_2}{4}\right)^{1/3} & \text{if } \beta_1 = 0 \\ 2\sqrt{\frac{-\beta_1}{6}} \sinh\left(\frac{1}{3}\text{arc sinh } C\right) & \text{if } \beta_1 < 0 \\ 2\sqrt{\frac{\beta_1}{6}} \text{sgn}(C) \cosh\left(\frac{1}{3}\text{arc cosh}(\text{sgn}(C)C)\right) & \text{if } \beta_1 > 0 \text{ and } |C| > 1 \\ 2\sqrt{\frac{\beta_1}{6}} \cos\left(\frac{4}{3}\pi + \frac{1}{3}\text{arc cos } C\right) & \text{if } \beta_1 > 0 \text{ and } |C| \leq 1, \end{cases}$$

where  $C = \beta_2 \left(\frac{3}{2|\beta_1|}\right)^{3/2}$ . The reason we choose  $4\pi/3$  rather than 0 or  $2\pi/3$  in the last line is for the consistency of the case  $\beta_2 = 0$ . However, in what follows, we will not use the explicit formula of  $\theta$ , we are going to work out only the estimate uniform in  $\theta$ . Because

$$\frac{-b(x)}{x - \theta} = 4(x - \theta)^2 + 12\theta(x - \theta) + 12\theta^2 - 2\beta_1 = 4(x + \theta/2)^2 + 3\theta^2 - 2\beta_1,$$

we obtain

$$\inf_{x > r} \frac{-b(x)}{x - \theta} = \begin{cases} 4(r + \theta/2)^2 + 3\theta^2 - 2\beta_1, & \text{if } r \geq -\theta/2 \\ 3\theta^2 - 2\beta_1, & \text{if } r \leq -\theta/2, \end{cases} \quad r \geq \theta.$$

$$\inf_{x < r} \frac{-b(x)}{x - \theta} = \begin{cases} 4(r + \theta/2)^2 + 3\theta^2 - 2\beta_1, & \text{if } r \leq -\theta/2 \\ 3\theta^2 - 2\beta_1, & \text{if } r \geq -\theta/2, \end{cases} \quad r < \theta.$$

Naturally, one may define  $K(r)$  as the right-hand sides, but then the computations for the lower bounds of  $\delta_{\pm}(K)$  become very complicated. Here, we adopt a simplification. Set  $r_{\theta} = r - \theta$ . Because

$$4(r + \theta/2)^2 + 3\theta^2 - 2\beta_1 = 12(\theta + r_{\theta}/2)^2 + r_{\theta}^2 - 2\beta_1 \geq r_{\theta}^2 - 2\beta_1, \\ 3\theta^2 - 2\beta_1 \geq 9\theta^2/4 - 2\beta_1,$$

when  $r \geq \theta$  (equivalently,  $r_{\theta} \geq 0$ ), we can choose

$$K(r) = K_{\theta}(r) = \begin{cases} r_{\theta}^2 - 2\beta_1, & \text{if } r_{\theta} > -3\theta/2 \\ 9\theta^2/4 - 2\beta_1, & \text{if } r_{\theta} < -3\theta/2. \end{cases}$$

By symmetry, one can define  $K(r)$  for the case of  $r \leq \theta$  as follows:

$$K(r) = \begin{cases} r_{\theta}^2 - 2\beta_1, & \text{if } r_{\theta} < -3\theta/2 \\ 9\theta^2/4 - 2\beta_1, & \text{if } r_{\theta} > -3\theta/2. \end{cases}$$

Obviously,  $K$  is a continuous piecewise  $C^1$ -function.

Suppose that  $\theta < 0$  for a moment. We use the notation  $G(r)$  defined in the proof (d) of Theorem 4.1. Since  $G(r)$  is continuous in  $r$ ,  $G(r)$  is equal to the constant  $K(-\theta/2)$  on  $(\theta, -\theta/2]$ , and  $K' > 0$  on  $(\theta, \infty)$ , we have  $\sup_{r>\theta} G(r) = \sup_{r\geq-\theta/2} G(r)$ . Clearly,  $\lim_{r\rightarrow\infty} K(r) \int_{\theta}^r (u-\theta)du = \infty$  and hence we can ignore (4.7) and handle with (4.8) only. There are two cases.

(a) Let  $K(-\theta/2) \int_{\theta}^{-\theta/2} (u-\theta)du < 1$ . That is  $9\theta^2/4 < \beta_1 + \sqrt{\beta_1^2 + 2}$ . In this case, the solution to (4.8) should satisfy  $r_+ - \theta > -3\theta/2$ . Solving equation

$$(r_{\theta}^2 - 2\beta_1) \int_{\theta}^r (u-\theta)du = 1, \quad r_{\theta} > -3\theta/2,$$

we get  $(r_+ - \theta)^2 = \beta_1 + \sqrt{\beta_1^2 + 2}$ . Then

$$\begin{aligned} -\frac{1}{2} \int_{\theta}^{r_+} (x-\theta)^2 K'(x)dx &= -\frac{1}{2} \int_{-3\theta/2}^{r_+-\theta} x^2 \cdot 2xdx \\ &= -\frac{1}{4}(r_+ - \theta)^4 + \frac{81}{64}\theta^4 \\ &= -\frac{1}{4}\left(\sqrt{\beta_1^2 + 2} + \beta_1\right)^2 + \frac{81}{64}\theta^4. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \sup_{r\geq-\theta/2} G(r) &= G(r_+) \geq \left(\sqrt{\beta_1^2 + 2} - \beta_1\right) \exp\left[-\frac{1}{4}\left(\sqrt{\beta_1^2 + 2} + \beta_1\right)^2 + \frac{81}{64}\theta^4\right] \\ &\geq \left(\sqrt{\beta_1^2 + 2} - \beta_1\right) \exp\left[-\frac{1}{4}\left(\sqrt{\beta_1^2 + 2} + \beta_1\right)^2\right] \\ &= \frac{\sqrt{\beta_1^2 + 2} - \beta_1}{\sqrt{e}} \exp\left[-\frac{1}{2}\beta_1\left(\beta_1 + \sqrt{\beta_1^2 + 2}\right)\right]. \end{aligned}$$

(b) Let  $K(-\theta/2) \int_{\theta}^{-\theta/2} (u-\theta)du \geq 1$ . Equivalently,  $9\theta^2/4 \geq \beta_1 + \sqrt{\beta_1^2 + 2}$ . In this case, the solution to (4.8) satisfies  $r_+ \in (\theta, -\theta/2)$ . Since  $K$  is a constant on  $(\theta, -\theta/2)$ , by (4.13) and (4.12),  $G = K$  on  $(\theta, -\theta/2]$ . Hence

$$\begin{aligned} \sup_{r>\theta} G(r) &= G(r_+) = K(-\theta/2) = \frac{9}{4}\theta^2 - 2\beta_1 \geq \sqrt{\beta_1^2 + 2} - \beta_1 \\ &\geq \left(\sqrt{\beta_1^2 + 2} - \beta_1\right) \exp\left[-\frac{1}{4}\left(\sqrt{\beta_1^2 + 2} + \beta_1\right)^2\right]. \end{aligned}$$

Combining (a) with (b) and (4.6), we obtain

$$\delta_+(K) \geq \frac{\sqrt{\beta_1^2 + 2} - \beta_1}{\sqrt{e}} \exp\left[-\frac{1}{2}\beta_1\left(\beta_1 + \sqrt{\beta_1^2 + 2}\right)\right].$$

Next, we estimate  $\delta_-(K)$ . Now,  $K(r) = r_\theta^2 - 2\beta_1$  on  $(-\infty, \theta)$  since  $\theta < 0$ . From (4.8), we get the same solution  $(r_- - \theta)^2 = \beta_1 + \sqrt{\beta_1^2 + 2}$ . But

$$\begin{aligned} -\frac{1}{2} \int_\theta^{r_-} (x - \theta)^2 K'(x) dx &= -\frac{1}{2} \int_0^{r_- - \theta} x^2 \cdot 2x dx \\ &= -\frac{1}{4} (r_- - \theta)^4 = -\frac{1}{4} \left( \sqrt{\beta_1^2 + 2} + \beta_1 \right)^2. \end{aligned}$$

By (4.6) again, we get

$$\delta_-(K) \geq \frac{\sqrt{\beta_1^2 + 2} - \beta_1}{\sqrt{e}} \exp \left[ -\frac{1}{2} \beta_1 \left( \beta_1 + \sqrt{\beta_1^2 + 2} \right) \right].$$

Therefore, we have proved the required lower bound in the case of  $\theta < 0$ .

By symmetry, the same conclusion holds when  $\theta > 0$ . The proof for  $\theta = 0$  is much simpler as shown below.

When  $\beta_2 = 0$ , we simply let  $\theta = 0$ . Then

$$\frac{-b(x)}{x} = 4x^2 - 2\beta_1, \quad x \neq 0.$$

We choose  $K(r) = 4r^2 - 2\beta_1$ . Then the equation (4.8) gives us

$$r_\pm^2 = \frac{1}{4} \left( \beta_1 + \sqrt{\beta_1^2 + 8} \right).$$

Because

$$\int_0^r \left[ \int_0^x u du \right] dK(x) = \int_0^r 4x^3 dx = r^4,$$

by (4.6), we obtain the last required assertion.  $\square$

We will improve the estimate of Example 4.3 in §5 (Example 5.3) by a different method.

Before moving further, let us make some remarks about the estimate given in Example 4.3. Recall that at the beginning of the proof, in choosing the function  $K(r)$ , the term  $12(\theta + r_\theta/2)^2$  was removed, this simplified greatly the proof since the original quartic equation is reduced to a quadratic one. For this reason one may worry lost too much in the estimation and we want to know the best estimate we can get by part (2) of Theorem 4.1. For this, we use a different trick. Consider the case of  $\theta < 0$  only. We use the complete form of  $K$ :

$$K(r) = \begin{cases} 4(r + \theta/2)^2 + 3\theta^2 - 2\beta_1, & \text{if } r \geq -\theta/2 \\ 3\theta^2 - 2\beta_1, & \text{if } \theta \leq r \leq -\theta/2 \\ 4(r + \theta/2)^2 + 3\theta^2 - 2\beta_1, & \text{if } r < \theta. \end{cases}$$

(i) Following the proof of Example 4.3, we study first the estimation of  $\delta_+(K)$ . There are two cases.

(a) Let  $K(-\theta/2) \int_{\theta}^{-\theta/2} (u - \theta) du < 1$ . That is,  $3\theta^2 < \beta_1 + \sqrt{\beta_1^2 + 8/3}$ . The idea is that in looking for a uniform estimate, we may regard  $r$  as a parameter rather than  $\theta$ . In other words, instead of solving equation (4.8)

$$(4r_{\theta}^2 + 12\theta r_{\theta} + 12\theta^2 - 2\beta_1) \int_{\theta}^r (u - \theta) du = 1, \quad r_{\theta} > -3\theta/2$$

in  $r$ , we solve the equation in  $\theta$ . Then the equation has two solutions:

$$\theta = \frac{1}{6} \left( -3r_{\theta} \pm \sqrt{6\beta_1 + 6/r_{\theta}^2 - 3r_{\theta}^2} \right).$$

Since  $\theta$  is real,  $r_{\theta}$  must satisfy

$$r_{\theta}^2 \leq \beta_1 + \sqrt{\beta_1^2 + 2}. \quad (4.14)$$

Next, in the “+” case,  $\theta < 0$  iff

$$r_{\theta}^2 > (\beta_1 + \sqrt{\beta_1^2 + 8})/4, \quad (4.15)$$

and it is obvious that  $r_{\theta} > -3\theta/2$ . In the “-” case, it is automatically that  $\theta < 0$  and  $r_{\theta} > 3\theta/2$  iff

$$r_{\theta}^2 > (3\beta_1 + \sqrt{9\beta_1^2 + 24})/4, \quad (4.16)$$

To estimate the decay exponent, note that on the one hand, we have

$$\begin{aligned} \theta r_{\theta}^3 &= \frac{1}{6} r_{\theta}^2 \left( -3r_{\theta}^2 \pm \sqrt{6\beta_1 r_{\theta}^2 + 6 - 3r_{\theta}^4} \right) \\ &= \frac{1}{6} z \left( -3z \pm \sqrt{6\beta_1 z + 6 - 3z^2} \right), \end{aligned}$$

where  $z = r_{\theta}^2$ . On the other hand, we have

$$-\frac{1}{2} \int_{-3\theta/2}^{r_{\theta} - \theta} x^2 K'(x) dx = -r_{\theta}^4 - 2\theta r_{\theta}^3 - \frac{27}{16} \theta^4.$$

Replacing  $r_{\theta}^2$  with  $z$  on the right-hand side plus some computation, we finally get

$$\begin{aligned} -\frac{1}{2} \int_{-3\theta/2}^{r_{\theta} - \theta} x^2 K'(x) dx &= -\frac{3}{64} \left[ -2z^2 + 8\beta_1 z + \beta_1^2 + 8 + \frac{2\beta_1}{z} + \frac{1}{z^2} \right] \\ &\quad \pm \frac{\sqrt{3}}{96} (9 + 9\beta_1 z - 23z^2) \sqrt{\frac{2\beta_1}{z} + \frac{2}{z^2} - 1}. \end{aligned}$$

To obtain the uniform lower bound, by (4.14) and (4.15), we need to minimize the right-hand side under the constrain

$$\begin{cases} (\beta_1 + \sqrt{\beta_1^2 + 8})/4 < z \leq \beta_1 + \sqrt{\beta_1^2 + 2} & \text{in the “+” case} \\ (3\beta_1 + \sqrt{9\beta_1^2 + 24})/4 < z \leq \beta_1 + \sqrt{\beta_1^2 + 2} & \text{in the “-” case.} \end{cases}$$

A numerical computation shows that the first case is smaller than the second one and its leading term is approximately  $-0.8\beta_1^2$ .

(b) Let  $K(-\theta/2) \int_{\theta}^{-\theta/2} (u-\theta) du \geq 1$ . That is,  $3\theta^2 \geq \beta_1 + \sqrt{\beta_1^2 + 8/3}$ . Then we have the lower bound  $\sqrt{\beta_1^2 + 8/3} - \beta_1$  which is decayed slowly than exponential.

(ii) Next, in the case of  $r < \theta$ , by assumption,  $\theta < 0$  and  $r_{\theta} < 0$ , we have only one solution

$$\theta = \frac{1}{6} \left( -3r_{\theta} - \sqrt{6\beta_1 + 6/r_{\theta}^2 - 3r_{\theta}^2} \right),$$

and furthermore  $\theta < 0$  iff

$$r_{\theta}^2 < \left( \beta_1 + \sqrt{\beta_1^2 + 8} \right) / 4.$$

To estimate the decay exponent, note that on the one hand, since  $r_{\theta} < 0$ , we have

$$\begin{aligned} \theta r_{\theta}^3 &= \frac{1}{6} r_{\theta}^2 \left( -3r_{\theta}^2 + \sqrt{6\beta_1 r_{\theta}^2 + 6 - 3r_{\theta}^4} \right) \\ &= \frac{1}{6} z \left( -3z + \sqrt{6\beta_1 z + 6 - 3z^2} \right). \end{aligned}$$

On the other hand, we have

$$-\frac{1}{2} \int_0^{r_{\theta} - \theta} x^2 K'(x) dx = -r_{\theta}^4 - 2\theta r_{\theta}^3.$$

Hence

$$-\frac{1}{2} \int_0^{r_{\theta} - \theta} x^2 K'(x) dx = -\frac{1}{\sqrt{3}} z \sqrt{2\beta_1 z + 2 - z^2}.$$

To obtain the uniform lower bound, it suffices to minimize the right-hand side under the constrain

$$0 < z < \left( \beta_1 + \sqrt{\beta_1^2 + 8} \right) / 4.$$

A numerical computation shows that the resulting estimate is bigger than  $-0.8\beta_1^2$ .

(iii) Finally, we conclude that the estimate on the exponent obtained so far is approximately  $-0.8\beta_1^2$ . Comparing this with our estimate  $-\beta_1^2$ , it is clear that there is no much room left for an improvement by part (2) of Theorem 4.1.

We now study the general criteria and estimates of  $\lambda_1(L)$  and  $\lambda_0^{\pm}(\theta)$  (see (4.17) below for definitions) in dimension one. For this, we need more notation.

Fix an arbitrary reference point  $\theta \in \mathbb{R}$ , not necessarily a root of  $b(x) = -u'(x)$ . Let  $\mathbb{R}_{\theta}^{+} = (\theta, \infty)$ ,  $\mathbb{R}_{\theta}^{-} = (-\infty, \theta)$ ,  $\overline{\mathbb{R}}_{\theta}^{+} = [\theta, \infty)$ , and  $\overline{\mathbb{R}}_{\theta}^{-} = (-\infty, \theta]$ . Recall that

$C_\theta(x) = \int_\theta^x b/a$ . Define

$$\begin{aligned}
\varphi_\theta(x) &= \int_\theta^x e^{-C}, \quad \delta_\theta^\pm = \sup_{x \in \mathbb{R}_\theta^\pm} \varphi(x) \int_x^{\pm\infty} \frac{e^C}{a}, \\
\mathcal{F}_{I\theta}^\pm &= \left\{ f \in C(\overline{\mathbb{R}_\theta^\pm}) \cap C^1(\mathbb{R}_\theta^\pm) : f(\theta) = 0, f'|_{\mathbb{R}_\theta^\pm} > 0 \right\}, \\
\mathcal{F}_{II\theta}^\pm &= \left\{ f \in C(\overline{\mathbb{R}_\theta^\pm}) : f(\theta) = 0, (\pm f)|_{\mathbb{R}_\theta^\pm} > 0 \right\}, \\
I_\theta^\pm(f)(x) &= \frac{e^{-C(x)}}{f'(x)} \int_x^{\pm\infty} \frac{f e^C}{a}, \quad \pm(x - \theta) \geq 0, \quad f \in \mathcal{F}_{I\theta}^\pm, \\
II_\theta^\pm(f)(x) &= \frac{1}{f(x)} \int_\theta^{\pm\infty} \varphi_\theta(x \wedge \cdot) \frac{f e^C}{a} = \frac{1}{f(x)} \int_\theta^x e^{-C(y)} dy \int_y^{\pm\infty} \frac{f e^C}{a}, \\
&\quad \pm(x - \theta) \geq 0, \quad f \in \mathcal{F}_{II\theta}^\pm, \\
\lambda_0^\pm(\theta) &= \inf \left\{ D(f) : f|_{\mathbb{R} \setminus \mathbb{R}_\theta^\pm} = 0, f \in C(\overline{\mathbb{R}_\theta^\pm}) \cap C^1(\mathbb{R}_\theta^\pm), \mu(f^2) = 1 \right\}. \quad (4.17)
\end{aligned}$$

**Theorem 4.4.** *The comparison of  $\lambda_1(L)$  and  $\lambda_0^\pm(\theta)$  and their estimates are given as follows.*

- (1)  $\inf_{\theta \in \mathbb{R}} [\lambda_0^+(\theta) \vee \lambda_0^-(\theta)] \geq \lambda_1(L) \geq \sup_{\theta \in \mathbb{R}} [\lambda_0^+(\theta) \wedge \lambda_0^-(\theta)]$ . In particular,  $\lambda_1(L) = \lambda_0^+(\bar{\theta})$ , where  $\bar{\theta}$  is the solution to the equation  $\lambda_0^+(\theta) = \lambda_0^-(\theta)$ ,  $\theta \in [-\infty, \infty]$ .
- (2) If  $m$  is the medium of  $\mu$ , then  $2[\lambda_0^+(m) \wedge \lambda_0^-(m)] \geq \lambda_1(L) \geq \lambda_0^+(m) \wedge \lambda_0^-(m)$ .
- (3)  $\lambda_0^\pm(\theta) \geq \sup_{f \in \mathcal{F}_{II\theta}^\pm} \inf_{x \in \mathbb{R}_\theta^\pm} II_\theta^\pm(f)(x)^{-1} \geq \sup_{f \in \mathcal{F}_{I\theta}^\pm} \inf_{x \in \mathbb{R}_\theta^\pm} I_\theta^\pm(f)(x)^{-1}$ . Moreover, the sign of equalities hold whenever both  $a$  and  $b$  are continuous.
- (4)  $(\delta_\theta^\pm)^{-1} \geq \lambda_0^\pm(\theta) \geq (4\delta_\theta^\pm)^{-1}$ .

*Proof.* The first assertion of part (1) is just [18; Theorem 3.3]. The lower bound in part (2) follows from the one of part (1). As remarked above [18; Theorem 3.3], from the proof of [18; Theorem 3.1], it follows that

$$\lambda_1(L) \leq \inf_{\theta \in \mathbb{R}} [\lambda_0^+(\theta)\mu(\theta, \infty)] \wedge [\lambda_0^-(\theta)\mu(-\infty, \theta)].$$

Hence, the upper bound in part (2) follows immediately. The variational formulas for the lower bounds given in part (3) is a copy of [19; Theorem 1.1]. In which, the corresponding variational formulas for the upper bounds are also presented, but omitted here. Part (4) was proven in [18; Theorem 1.1]. From these quoted papers, one can find some more sharper estimates and further references.

It remains to prove the second assertion of part (1). For this, it suffices to show that  $\lambda_0^\pm(\theta)$  is continuous in  $\theta$ . By symmetry, it is enough to prove that  $\lambda_0^+(\theta)$  is continuous in  $\theta$ . Let  $\theta_1 < \theta_2 < \infty$ . Clearly,  $\lambda_0^+(\theta_1) < \lambda_0^+(\theta_2)$ . Given  $\varepsilon \in (0, 1)$ , choose  $f = f_\varepsilon \in C^1(\theta_1, \infty) \cap C[\theta_1, \infty)$  such that  $f(\theta_1) = 0$ ,  $\int_{\theta_1}^\infty f^2 d\mu = 1$  and  $A - \varepsilon \leq \lambda_0^+(\theta_1)$ , where  $A = A_\varepsilon = \int_{\theta_1}^\infty f'^2 d\mu$ . By the continuity of  $f$ , when

$\theta_2 - \theta_1 > 0$  is sufficient small, we have

$$\begin{aligned} & \left| f(\theta_2)^2 \int_{\theta_2}^{\infty} d\mu - 2f(\theta_2) \int_{\theta_2}^{\infty} f d\mu - \int_{\theta_1}^{\theta_2} f^2 d\mu \right| \\ & \leq f(\theta_2)^2 \int_{\theta_1}^{\infty} d\mu + 2f(\theta_2) + \int_{\theta_1}^{\theta_2} f^2 d\mu \\ & < \varepsilon. \end{aligned}$$

Then  $\int_{\theta_2}^{\infty} [f - f(\theta_2)]^2 d\mu > 1 - \varepsilon$  and furthermore

$$\lambda_0^+(\theta_2) \leq \int_{\theta_2}^{\infty} f'^2 d\mu / \int_{\theta_2}^{\infty} [f - f(\theta_2)]^2 d\mu \leq \frac{A}{1 - \varepsilon} \leq \frac{\lambda_0^+(\theta_1) + \varepsilon}{1 - \varepsilon}.$$

Since  $\varepsilon$  can be arbitrarily small, we obtain the required assertion.  $\square$

As an illustration of the applications of Theorem 4.4, we discuss Examples 4.2 and 4.3 again.

**Example 4.5.** *Everything in premise is the same as in Example 4.2. We have*

$$\frac{2\alpha}{\delta} \geq \lambda_1(L_{\alpha,\beta}) \geq \frac{\alpha}{4\delta},$$

where

$$\delta = \sup_{x>0} \int_0^x e^{y^2} dy \int_x^{\infty} e^{-y^2} dy \approx 0.239405.$$

*Proof.* First, we have the root  $\theta = -\beta/(2\alpha)$  of  $u'(x)$ , it is also the medium of the measure. Next,

$$C_{\theta}(x) = -\alpha(x-\theta)^2, \quad \varphi_{\theta}(x) = \int_0^{x-\theta} e^{\alpha y^2} dy, \quad \int_x^{\infty} e^{-\alpha(y-\theta)^2} dy = \int_{x-\theta}^{\infty} e^{-\alpha y^2} dy.$$

Hence  $\delta_{\theta}^+ = \delta/\alpha$ . By symmetry, we also have  $\delta_{\theta}^- = \delta/\alpha$ . The assertion now follows from parts (2) and (4) of Theorem 4.4.  $\square$

**Example 4.6.** *Everything in premise is the same as in Example 4.3. We have*

- (1)  $\lim_{|\beta_2| \rightarrow \infty} \lambda_1(L_{\beta_1, \beta_2}) = \infty$ .
- (2) For  $\beta_1 \geq 0$ , we have

$$\lambda_1(L_{\beta_1, 0}) \leq 4e^{14} \exp \left[ -\frac{1}{4}\beta_1^2 + 2\log(1 + \beta_1) \right].$$

*Proof.* By symmetry of  $u(x)$  in  $x$ , one may assume that  $\beta_2 \geq 0$ . Let  $\theta$  be a real root of  $u'(x)$ . Clearly,  $\lim_{\beta_2 \rightarrow \infty} \theta = -\infty$ . Moreover,  $u(x) - u(\theta) = (x - \theta)^2[(x - \theta)^2 + 4\theta(x - \theta) + 6\theta^2 - \beta_1]$ . Hence

$$\begin{aligned}
& \int_{\theta}^x e^{u(y)} dy \int_x^{\infty} e^{-u(z)} dz \\
&= \int_{\theta}^x e^{u(y)-u(\theta)} dy \int_x^{\infty} e^{-u(z)+u(\theta)} dz \\
&= \int_0^{x-\theta} e^{y^2(y^2+4\theta y+6\theta^2-\beta_1)} dy \int_{x-\theta}^{\infty} e^{-z^2(z^2+4\theta z+6\theta^2-\beta_1)} dz \\
&= \int_0^{x-\theta} dy \int_{x-\theta}^{\infty} \exp \left[ -(z^2-y^2) \left( z^2+y^2+4\theta \left( z+\frac{y^2}{z+y} \right) + 6\theta^2 - \beta_1 \right) \right] dz.
\end{aligned} \tag{4.18}$$

(a) We now prove the first assertion. It says that the parameter  $\beta_2$  plays a role for  $\lambda_1(L_{\beta_1, \beta_2})$ , in contrast with Example 4.5. For  $x \geq \theta$ , by (4.18), we have

$$\begin{aligned}
& \int_{\theta}^x e^{u(y)} dy \int_x^{\infty} e^{-u(z)} dz \\
&= \int_0^{x-\theta} dy \int_{x-\theta}^{\infty} dz \exp \left[ -(z^2-y^2) \left[ (z+2\theta)^2 + \left( y + \frac{2\theta y}{z+y} \right)^2 - 4\theta^2 \left( \frac{y}{z+y} \right)^2 + 2\theta^2 - \beta_1 \right] \right] \\
&\leq \int_0^{x-\theta} dy \int_{x-\theta}^{\infty} dz \exp \left[ -(z^2-y^2) \left[ -4\theta^2 \left( \frac{y}{z+y} \right)^2 + 2\theta^2 - \beta_1 \right] \right].
\end{aligned}$$

Since  $z \geq y \geq 0$ , we have  $y/(z+y) \leq 1/2$ . The right-hand side is controlled by

$$\int_0^{x-\theta} dy \int_{x-\theta}^{\infty} e^{-(z^2-y^2)(\theta^2-\beta_1)} dz, \quad x \geq \theta. \tag{4.19}$$

We now use Conte's estimate (cf. [20]):

$$x \left( 1 + \frac{x^2}{12} \right) e^{-3x^2/4} < e^{-x^2} \int_0^x e^{y^2} dy \leq \frac{\pi^2}{8x} (1 - e^{-x^2}), \quad x > 0$$

and Gautschi's estimate (cf. [21]):

$$\frac{1}{2} \left[ (x^p + 2)^{1/p} - x \right] < e^{x^p} \int_x^{\infty} e^{-y^p} dy \leq C_p \left[ \left( x^p + \frac{1}{C_p} \right)^{1/p} - x \right], \quad x \geq 0,$$

$$C_p := \Gamma(1 + 1/p)^{p/(p-1)}, \quad p > 1; \quad C_2 = \pi/4.$$

Thus,

$$\begin{aligned}
\int_0^x e^{cy^2} dy \int_x^{\infty} e^{-cz^2} dz &\leq \frac{\pi^2}{8c\sqrt{c}x} (1 - e^{-cx^2}) \cdot \frac{\pi}{4} \left( \sqrt{cx^2 + \frac{4}{\pi}} - \sqrt{c}x \right) \\
&\leq \frac{\pi^2}{8c\sqrt{c}x} \sqrt{\frac{\pi}{4}} (1 - e^{-cx^2}), \quad x \geq 0.
\end{aligned}$$



Noting that  $(1 - e^{-cx^2})/x \leq cx \leq c$  for all  $x \in (0, 1]$  and  $(1 - e^{-cx^2})/x \leq 1/x \leq 1$  for all  $x \geq 1$ , we obtain

$$\int_0^x e^{cy^2} dy \int_x^\infty e^{-cz^2} dz \leq \frac{\pi^{5/2}}{16\sqrt{c}}, \quad x \geq 0, \quad c \geq 1.$$

Therefore

$$\begin{aligned} \delta_\theta^+ &= \sup_{x > \theta} \int_\theta^x e^{u(y)} dy \int_x^\infty e^{-u(z)} dz \\ &\leq \sup_{x > 0} \int_0^x dy \int_x^\infty e^{-(z^2 - y^2)(\theta^2 - \beta_1)} dz \\ &\leq \frac{\pi^{5/2}}{16\sqrt{\theta^2 - \beta_1}} \rightarrow 0 \quad \text{as } \theta \rightarrow -\infty. \end{aligned}$$

For  $\delta_\theta^-$ , the proof is similar. As an analogue of (4.18), we have

$$\begin{aligned} &\int_x^\theta e^{u(y)} dy \int_{-\infty}^x e^{-u(z)} dz \\ &= \int_{x-\theta}^0 dy \int_{-\infty}^{x-\theta} dz \exp \left[ - (z^2 - y^2) \left[ (z + 2\theta)^2 + \left( y + \frac{2\theta y}{z + y} \right)^2 \right. \right. \\ &\quad \left. \left. - 4\theta^2 \left( \frac{y}{z + y} \right)^2 + 2\theta^2 - \beta_1 \right] \right]. \end{aligned}$$

Since  $z \leq y \leq 0$ , we have  $|y/(z + y)| \leq 1/2$ , we obtain

$$\int_x^\theta e^{u(y)} dy \int_{-\infty}^x e^{-u(z)} dz \leq \int_{x-\theta}^0 dy \int_{-\infty}^{x-\theta} e^{-(z^2 - y^2)(\theta^2 - \beta_1)} dz, \quad x \leq \theta.$$

We have thus returned to (4.19).

Now, the first assertion follows from parts (1) and (4) of Theorem 4.4.

(b) For the upper bound in part (2), since  $\beta_2 = 0$ , we have  $\theta = 0$ . We need to show that

$$\sup_{x > 0} \int_0^x e^{y^4 - \beta_1 y^2} dy \int_x^\infty e^{-z^4 + \beta_1 z^2} dz \geq \frac{1}{4e^{14}} \exp \left[ \frac{1}{4} \beta_1^2 - 2 \log(1 + \beta_1) \right].$$

Since

$$\begin{aligned} &\int_0^x e^{y^4 - \beta_1 y^2} dy \int_x^\infty e^{-z^4 + \beta_1 z^2} dz \\ &= \frac{1}{4} \int_{-\beta_1/2}^{x^2 - \beta_1/2} \frac{e^{y^2}}{\sqrt{y + \beta_1/2}} dy \int_{x^2 - \beta_1/2}^\infty \frac{e^{-z^2}}{\sqrt{z + \beta_1/2}} dz \\ &> \frac{1}{4} \int_{-\beta_1/2}^{x^2 - \beta_1/2} \frac{e^{y^2}}{\sqrt{y + \beta_1/2}} dy \int_{x^2 - \beta_1/2}^{\beta_1/2} \frac{e^{-z^2}}{\sqrt{z + \beta_1/2}} dz, \end{aligned}$$

when  $\beta_1 \geq 1$ , we have

$$\begin{aligned} & \int_{-\beta_1/2}^{1-\beta_1/2} \frac{e^{y^2}}{\sqrt{y+\beta_1/2}} dy \int_{1-\beta_1/2}^{\beta_1/2} \frac{e^{-z^2}}{\sqrt{z+\beta_1/2}} dz \\ & \geq \frac{1}{\beta_1} \int_{-\beta_1/2}^{1-\beta_1/2} e^{y^2} dy \int_{1-\beta_1/2}^{\beta_1/2} e^{-z^2} dz. \end{aligned}$$

It suffices to show that

$$\frac{1}{\beta_1} \int_{-\beta_1/2}^{1-\beta_1/2} e^{y^2} dy \int_{1-\beta_1/2}^{\beta_1/2} e^{-z^2} dz \geq \frac{1}{e^{14}} \exp \left[ \frac{1}{4} \beta_1^2 - 2 \log(1 + \beta_1) \right],$$

or

$$\int_{-\beta_1/2}^{1-\beta_1/2} e^{y^2} dy \int_{1-\beta_1/2}^{\beta_1/2} e^{-z^2} dz \geq \exp \left[ \frac{1}{4} \beta_1^2 - \log(1 + \beta_1) - 14 \right].$$

Since

$$\begin{aligned} & \int_{1-\beta_1/2}^{\beta_1/2} e^{-z^2} dz \rightarrow \int_{-\infty}^{\infty} e^{-z^2} dz < \infty, \\ & \int_{-\beta_1/2}^{1-\beta_1/2} e^{y^2} dy = \int_{\beta_1/2-1}^{\beta_1/2} e^{y^2} dy \geq \exp \left[ \left( \frac{\beta_1}{2} - 1 \right)^2 \right] \rightarrow \infty, \\ & \frac{\int_{-\beta_1/2}^{1-\beta_1/2} e^{y^2} dy}{\exp[\beta_1^2/4 - \log \beta_1]} \sim \frac{\exp[\beta_1^2/4] - \exp[(1 - \beta_1/2)^2]}{\exp[\beta_1^2/4]} \\ & \sim 1 - e^{1-\beta_1} \\ & \sim 1 \quad \text{as } \beta_1 \rightarrow \infty, \end{aligned}$$

it is easy to check first that

$$\log \left[ \int_{-\beta_1/2}^{1-\beta_1/2} e^{y^2} dy \int_{1-\beta_1/2}^{\beta_1/2} e^{-z^2} dz \right] \geq \frac{1}{4} \beta_1^2 - \log(1 + \beta_1) - 14$$

for  $\beta_1 \geq 1$  and then the required assertion for  $\beta_1 \geq 0$  by using mathematical softwares.  $\square$

Before moving further, let us study the lower bounds of  $\inf_{\beta_2 \geq 0} \lambda_1(L_{\beta_1, \beta_2})$  in terms of  $\delta_\theta^\pm$ . For this, we return to (4.18). Because

$$4\theta \left( z + \frac{y^2}{z+y} \right) + 6\theta^2 = 6 \left[ \theta + \frac{1}{3} \left( z + \frac{y^2}{z+y} \right) \right]^2 - \frac{2}{3} \left( z + \frac{y^2}{z+y} \right)^2 \geq -\frac{2}{3} (z + y/2)^2,$$

and so

$$\begin{aligned} z^2 + y^2 + 4\theta \left( z + \frac{y^2}{z+y} \right) + 6\theta^2 - \beta_1 & \geq z^2 + y^2 - \frac{2}{3} (z + y/2)^2 - \beta_1 \\ & = \frac{1}{6} (2z^2 - 4zy + 5y^2) - \beta_1 \\ & \geq \frac{1}{6} (z^2 + y^2) - \beta_1, \end{aligned}$$

it follows that

$$\begin{aligned}
& \int_0^x dy \int_x^\infty \exp \left[ - (z^2 - y^2) \left( z^2 + y^2 + 4\theta \left( z + \frac{y^2}{z+y} \right) + 6\theta^2 - \beta_1 \right) \right] dz \\
& \leq \int_0^x dy \int_x^\infty dz e^{-(z^2 - y^2)((z^2 + y^2)/6 - \beta_1)} \\
& = \int_0^x dy e^{y^4/6 - \beta_1 y^2} \int_x^\infty e^{-z^4/6 + \beta_1 z^2} dz.
\end{aligned} \tag{4.20}$$

Combining (4.18) with (4.20), we obtain

$$\delta_\theta^+ \leq \sup_{x>0} \int_0^x dy e^{y^4/6 - \beta_1 y^2} \int_x^\infty e^{-z^4/6 + \beta_1 z^2} dz.$$

The same upper bound holds for  $\delta_\theta^-$ . By parts (1) and (4) of Theorem 4.4, we obtain a lower estimate of  $\inf_{\beta_2 \geq 0} \lambda_1(L_{\beta_1, \beta_2})$ . However, the resulting bound is smaller than those given in Example 4.3.

We mention that the lower bound given in Example 4.3 may still be improved by applying part (3) of Theorem 4.3 to the test functions  $f_\pm$  constructed in the proof of Theorem 4.1. This observation is due to [22]. The proof is quite easy. Let for instance

$$- \sup_{x \in (\theta, \infty)} \frac{af_+'' + bf_+'}{f_+}(x) \geq \delta > 0.$$

Then  $f_+ \leq -(af_+'' + bf_+)/\delta$ . Noting that  $(e^C f_+')' = e^C (af_+'' + bf_+)/a$ , we obtain

$$\begin{aligned}
I_\theta^+(f_+)(x) &= \frac{e^{-C(x)}}{f_+'(x)} \int_x^\infty \frac{f_+ e^C}{a} \\
&\leq \frac{1}{\delta} \frac{e^{-C(x)}}{f_+'(x)} \int_x^\infty \left( - \frac{af_+'' + bf_+'}{a} \right) e^C \\
&= \frac{1}{\delta} \frac{e^{-C(x)}}{f_+'(x)} \int_x^\infty (-e^C f_+')' \\
&\leq \frac{1}{\delta} \frac{e^{-C(x)}}{f_+'(x)} e^{C(x)} f_+'(x) \\
&= \frac{1}{\delta}, \quad x > \theta.
\end{aligned}$$

Alternatively, one may apply the approximation procedure given in [19] to improve the lower bound. However, all the computations are quite complicated, and so we do not want to go further along this line.

We remark that the process in Example 4.6 (Example 4.3) possesses much stronger ergodic properties.

**Proposition 4.7.** *The processes corresponding to Example 4.3 is not only exponentially ergodic but also strongly ergodic. It has the empty essential spectrum. It satisfies the logarithmic Sobolev inequality but not the Nash (Sobolev) inequality.*

*Proof.* One may use the criteria given in [13; §5.4] to justify these assertions. For the reader's convenience, here we mention three criteria as follows. By the symmetry, we need only to write down the conditions on the half-line  $[0, \infty)$ .

*Logarithmic Sobolev inequality:*

$$\sup_{x>0} \left( \int_x^\infty e^{-u} \right) \left( \log \int_x^\infty e^{-u} \right) \int_0^x e^u < \infty.$$

*Strong ergodicity:*

$$\int_0^\infty dx e^{u(x)} \int_x^\infty e^{-u} < \infty.$$

*Nash (Sobolev) inequality:*

$$\sup_{x>0} \left( \int_x^\infty e^{-u} \right)^{1-2/\nu} \int_0^x e^u < \infty, \quad \nu > 2.$$

The second condition holds since

$$\frac{\int_x^\infty e^{-u}}{x^{-2}e^{-u}} \sim \frac{1}{2x^{-3} + x^{-2}u'} \sim \frac{x^3}{xu'} \rightarrow 0, \quad x \rightarrow \infty.$$

However, replacing  $x^{-2}$  with  $x^{-1}$  at the beginning, the same proof shows that the standard Ornstein-Uhlenbeck process is not strongly ergodic. For the third condition, note that  $\int_x^\infty e^{-u}$  and  $\int_0^x e^u$  have the leading order  $e^{-u}$  and  $e^u$  respectively. Hence the leading order of

$$\left( \int_x^\infty e^{-u} \right)^{1-2/\nu} \int_0^x e^u$$

is  $e^{2u/\nu} \rightarrow \infty$  as  $x \rightarrow \infty$ . Similarly, one can check the first condition. Alternatively, to see that the logarithmic Sobolev inequality holds, simply use the fact that  $\lim_{|x| \rightarrow \infty} u''(x) > 0$  (see [23]). We will come back to this point in Example 5.3. Finally, the logarithmic Sobolev inequality implies the essential spectrum to be empty.  $\square$

Finally, we study a perturbation of  $\lambda_1(L)$ .

**Proposition 4.8.** *Let  $a(x) \equiv 1$  and assume that  $\delta_\theta^\pm < \infty$  for some  $\theta \in \mathbb{R}$ . Next, let  $h$  satisfy  $\int_{\mathbb{R}} e^{C+h} < \infty$ . Define  $\delta_\theta^\pm(h) = \sup_{x \in \mathbb{R}_\theta^\pm} \int_\theta^x e^{-C-h} \int_x^{\pm\infty} e^{C+h}$ . If there exist constants  $K_1^\pm, \dots, K_4^\pm$  such that*

$$\pm \int_x^{\pm\infty} e^C \leq K_1^\pm e^{C(x)}, \quad \pm(x - \theta) \geq 0, \quad (4.21)$$

$$\pm \int_\theta^x e^{-C} \leq K_2^\pm e^{-C(x)}, \quad \pm(x - \theta) \geq 0, \quad (4.22)$$

$$\pm \int_x^{\pm\infty} e^C |e^h - 1| \leq K_3^\pm e^{C(x)}, \quad \pm(x - \theta) \geq 0, \quad (4.23)$$

$$\pm \int_\theta^x e^{-C} |e^{-h} - 1| \leq K_4^\pm e^{-C(x)}, \quad \pm(x - \theta) \geq 0, \quad (4.24)$$

then

$$\delta_\theta^\pm(h) \leq \delta_\theta^\pm + K_2^\pm K_3^\pm + K_1^\pm K_4^\pm + K_4^\pm K_3^\pm < \infty.$$

*Proof.* Here, we consider  $\delta_\theta^+(h)$  only. As in [5], we have

$$\begin{aligned} & \int_\theta^x e^{-C-h} \int_x^\infty e^{C+h} \\ &= \left[ \int_\theta^x e^{-C} + \int_\theta^x e^{-C}(e^{-h} - 1) \right] \cdot \left[ \int_x^\infty e^C + \int_x^\infty e^C(e^h - 1) \right] \\ &= \int_\theta^x e^{-C} \int_x^\infty e^C + \int_\theta^x e^{-C} \int_x^\infty e^C(e^h - 1) \\ &\quad + \int_\theta^x e^{-C}(e^{-h} - 1) \int_x^\infty e^C + \int_\theta^x e^{-C}(e^{-h} - 1) \int_x^\infty e^C(e^h - 1) \\ &\leq \delta_\theta^+ + K_2^+ K_3^+ + K_1^+ K_4^+ + K_4^+ K_3^+ < \infty. \quad \square \end{aligned}$$

The above result is a revised version of [5; Theorem 3.4], where instead of (4.23) and (4.24), the conditions

- (i)  $C(x)$  is strictly uniformly convex up to a bounded function
- (ii)  $\int_{\mathbb{R}} (e^{|h|} - 1) < \infty$

are employed. It is easy to check that these conditions together are stronger than (4.23) and (4.24). Clearly, under (4.21) and (4.22), conditions (4.23) and (4.24) are automatic for bounded  $h$ , for which, the condition (ii) here may fail.

**Example 4.9.** Let  $a(x) \equiv 1$  and  $C_\beta(x) = -x^4 + \beta x^2$ . Then  $\lambda_1(L_\beta) > 0$  for all  $\beta \in \mathbb{R}$ .

*Proof.* The case of  $\beta < 0$  is easy since  $-C_\beta$  is convex. Hence we assume that  $\beta \geq 0$ . Then  $-C_\beta$  is convex for large enough  $x$  and so the conclusion is known. Here we check it by using Proposition 4.8. Take  $C(x) = -x^4$  and regard  $h(x) = \beta x^2$  as a perturbation of  $C(x)$ . Clearly,  $\int_{\mathbb{R}} (e^{|h|} - 1) = \infty$ . Set  $\theta = 0$ .

First, by Gautschi's estimate, we have

$$e^{-C(x)} \int_x^\infty e^C = e^{x^4} \int_x^\infty e^{-y^4} dy \leq C_4 \left[ \left( x^4 + \frac{1}{C_4} \right)^{1/4} - x \right] \leq \Gamma\left(\frac{5}{4}\right) \approx 0.9064$$

for all  $x > 0$ . Next, we have

$$\begin{aligned} e^{C(x)} \int_x^\infty e^{-C} |e^{-h} - 1| &= e^{-x^4} \int_x^\infty e^{y^4} |e^{-\beta y^2} - 1| dy \\ &\leq e^{C(x)} \int_x^\infty e^{-C} \\ &= e^{-x^4} \int_x^\infty e^{y^4} dy \\ &< 0.6, \quad x > 0. \end{aligned}$$

Moreover,

$$e^{-C(x)} \int_x^\infty e^C |e^{-h} - 1| < e^{-C(x)} \int_x^\infty e^{C+h} \leq e^{\beta(0.7\beta+26)}, \quad x > 0.$$

By symmetry, the same estimates hold on  $(-\infty, 0]$ . Now, by Proposition 4.8 and Theorem 4.4, it follows that the leading order of the lower estimate of  $\lambda_1(L_\beta)$  is  $\exp[-0.7\beta^2]$  which is not far away from the optimal one:  $\exp[-\beta^2/4]$ .  $\square$

### 5. Logarithmic Sobolev inequality.

We begin this section with a result taken from [23; Corollary 1.4].

**Lemma 5.1.** *Let  $L = \Delta - \langle \nabla U, \nabla \rangle$  in  $\mathbb{R}^n$  and define  $\gamma(r) = \inf_{|x| \geq r} \lambda_{\min}(\text{Hess}(U)(x))$ . If  $\sup_{r \geq 0} \gamma(r) > 0$ , then we have*

$$\sigma(L) \geq \frac{2e}{a_0^2} \exp \left[ - \int_0^{a_0} r \gamma(r) dr \right] > 0,$$

where  $a_0 > 0$  is the unique solution to the equation  $\int_0^a \gamma(r) dr = 2/a$ .

This lemma says that the logarithmic Sobolev constant is positive whenever so is  $\lambda_{\min}(\text{Hess}(U)(x))$  at infinity. Unfortunately, as shown by Example 2.5, our models do not satisfy this condition even in the two-dimensional case. Hence, we justify the power of the estimate provided by the lemma only in dimensional one (compare with the criterion for the inequality, see for instance [13; Theorem 7.4]).

**Example 5.2.** *For Example 4.2, we have  $\lambda_1(L_{\alpha, \beta}) \geq \sigma(L_{\alpha, \beta}) \geq 2\alpha$  which are exact.*

*Proof.* Because  $u(x) = \alpha x^2 + \beta x$ , we have  $u''(x) = 2\alpha$  and so

$$\gamma(r) = \inf_{|x| \geq r} u''(x) = 2\alpha.$$

Next, since  $\int_0^a \gamma(r) dr = 2\alpha a$ . The unique solution to the equation

$$\int_0^a \gamma(r) dr = \frac{2}{a}$$

is  $a_0^2 = 1/\alpha$ . Noticing that  $\int_0^a r \gamma(r) dr = \alpha a^2$ , by Lemma 5.1, we obtain

$$\sigma(L_{\alpha, \beta}) \geq \frac{2e}{a_0^2} \exp \left[ - \alpha a_0^2 \right] = 2\alpha.$$

This is clearly exact since the well-known fact  $\lambda_1(L_{\alpha, \beta}) \geq \sigma(L_{\alpha, \beta})$  (cf. [13; Theorem 8.7]) and Example 4.2.  $\square$

**Example 5.3.** For Example 4.3, we have

$$\begin{aligned} \inf_{\beta_2} \lambda_1(L_{\beta_1, \beta_2}) &\geq \inf_{\beta_2} \sigma(L_{\beta_1, \beta_2}) \geq \frac{\sqrt{\beta_1^2 + 8} - \beta_1}{\sqrt{e}} \exp \left[ -\frac{1}{8} \beta_1 (\beta_1 + \sqrt{\beta_1^2 + 8}) \right] \\ &\geq \begin{cases} -2\beta_1 + \frac{2}{\sqrt{e/2} - \beta_1}, & \text{if } \beta_1 < 0 \\ 2\sqrt{2/e}, & \text{if } \beta_1 = 0 \\ \frac{1}{\sqrt{e/8} + \beta_1} \exp[-\beta_1^2/4], & \text{if } \beta_1 > 0 \end{cases} \end{aligned}$$

*Proof.* Because  $u(x) = x^4 - \beta_1 x^2 + \beta_2 x$ , we have  $u''(x) = 12x^2 - 2\beta_1$  and  $\gamma(r) = \inf_{|x| \geq r} u''(x) = 12r^2 - 2\beta_1$ . Next, since  $\int_0^a \gamma(r) dr = 4a^3 - 2\beta_1 a$ , the solution to the equation  $\int_0^a \gamma(r) dr = 2/a$  is as follows

$$a_0^2 = \frac{\beta_1 + \sqrt{\beta_1^2 + 8}}{4}.$$

Next, since

$$\int_0^a r \gamma(r) dr = a^2(3a^2 - \beta_1),$$

by Lemma 5.1, we obtain

$$\begin{aligned} \sigma(L_{\beta_1, \beta_2}) &\geq \frac{2e}{a_0^2} \exp[-a_0^2(3a_0^2 - \beta_1)] \\ &= \frac{\sqrt{\beta_1^2 + 8} - \beta_1}{\sqrt{e}} \exp \left[ -\frac{1}{8} \beta_1 (\beta_1 + \sqrt{\beta_1^2 + 8}) \right]. \quad \square \end{aligned}$$

Note that in the case of  $\beta_1 < 0$ , the Bakry-Emery criterion (cf. (2.9)) is available and gives us the lower bound  $-2\beta_1$  which is smaller than the estimate above. Example 5.3 is somehow unexpect since it improves Example 4.3 (In the special case that  $\beta_2 = 0$ , they are coincided). The reason is due to the fact that only the uniform estimate is treated in Example 4.3 and the linear term of  $U$  is ruled out in Lemma 5.1 (but the universal estimates depend on the linear term, cf. [13; Theorem 7.4]). Otherwise, the two methods may not be comparable in view of part (1) of Example 4.6. As mentioned in [23; Example 1.12] that the bounded perturbations should be carefully treated before applying Lemma 5.1.

*Proof of Proposition 1.4.* Let  $\beta_1 \geq 0$ . Note that

$$\sqrt{1 + \frac{8}{\beta_1^2}} \leq 1 + \frac{4}{\beta_1^2}.$$

We have  $\sqrt{\beta_1^2 + 8} \leq \beta_1 + 4/\beta_1$ . Hence

$$\exp \left[ -\frac{1}{8} \beta_1 (\beta_1 + \sqrt{\beta_1^2 + 8}) \right] \geq \frac{1}{\sqrt{e}} \exp \left[ -\frac{1}{4} \beta_1^2 \right].$$

Similarly, we have

$$\sqrt{\beta_1^2 + 8} - \beta_1 = \frac{8}{\sqrt{\beta_1^2 + 8} + \beta_1} \geq \frac{4}{\beta_1 + 2\beta_1^{-1}}.$$

By Example 5.3, we obtain  $\inf_{\beta_2} \sigma(L_{\beta_1, \beta_2}) \geq \exp[-\beta_1^2/4 - \log(1 + \beta_1)]$  for  $\beta_1 \geq 2$ . Combining this with Example 4.6, we get the required assertion.  $\square$

## 6. Continuous spin systems.

We begin this section with the ergodicity of our models in the finite dimensions. Consider the particle system on  $\Lambda$  with periodic boundary. Then the generator is

$$L_\Lambda = \Delta + \langle b, \nabla \rangle$$

where

$$b_i(x) = -u'(x_i) - 2J \sum_{j \in N(i)} (x_i - x_j)$$

for some  $u \in C^\infty(\mathbb{R})$ , constant  $J$ , and  $N(i)$  is the nearest neighbors of  $i$ . For simplicity, assume that  $J \geq 0$ , but it is not essential in this section. Recall that for the coupling by reflection, the coupling operator  $\bar{L}$  has the coefficients

$$a(x, y) = \begin{pmatrix} I & I - 2\bar{u}\bar{u}^* \\ I - 2\bar{u}\bar{u}^* & I \end{pmatrix}, \quad b(x, y) = \begin{pmatrix} b(x) \\ b(y) \end{pmatrix},$$

where  $\bar{u} = \bar{u}(x, y) = (x - y)/|x - y|$ . Furthermore, for  $f \in C[0, \infty) \cap C^2(0, \infty)$ , we have

$$\bar{L}f(|x - y|) = 4f''(|x - y|) + \frac{\langle x - y, b(x) - b(y) \rangle}{|x - y|} f'(|x - y|), \quad x \neq y$$

(cf. [13; Theorem 2.30]). To illustrate the idea, we restrict ourselves to the second model.

**Theorem 6.1.** *Let  $u(x_i) = x_i^4 - \beta x_i^2$  for all  $i \in \Lambda$ . Then the process is exponentially ergodic for any finite  $\Lambda$ . Moreover, the coupling by reflection  $(X_t, Y_t)$  gives us*

$$\bar{\mathbb{E}}^{x, y} f(|X_t - Y_t|) \leq f(|x - y|) e^{-\varepsilon t}, \quad t \geq 0,$$

where

$$\begin{aligned} f(r) &= \int_0^r e^{-C(s)} ds \int_s^\infty e^C \sqrt{\varphi}, \quad r > 0, \\ C(r) &= -\frac{1}{16|\Lambda|} r^4 + \frac{\beta}{4} r^2, \quad \varphi(r) = \int_0^r e^{-C}, \\ \varepsilon &= \varepsilon(\Lambda, \beta) = 4 \inf_{r > 0} \frac{\sqrt{\varphi(r)}}{f(r)} > 0. \end{aligned}$$



*Proof.* Because  $u'(x_i) = 4x_i^3 - 2\beta x_i$  and

$$b_i(x) = -4x_i^3 + 2\beta x_i - 2J \sum_{j \in N(i)} (x_i - x_j)$$

for all  $i$ . Thus,

$$b_i(x) - b_i(y) = -4(x_i^3 - y_i^3) + 2\beta(x_i - y_i) - 2J \sum_{j \in N(i)} (x_i - y_i - x_j + y_j).$$

Hence

$$\begin{aligned} \langle x - y, b(x) - b(y) \rangle &= -4 \sum_i (x_i - y_i)^2 (x_i^2 + x_i y_i + y_i^2) + 2\beta \sum_i (x_i - y_i)^2 \\ &\quad - J \sum_i \sum_{j \in N(i)} (x_i - y_i - x_j + y_j)^2 \\ &\leq - \sum_i (x_i - y_i)^4 + 2\beta \sum_i (x_i - y_i)^2 \\ &\leq -|\Lambda|^{-1} |x - y|^4 + 2\beta |x - y|^2, \end{aligned}$$

where  $|\Lambda|$  is the cardinality of  $\Lambda$ . It follows that

$$\frac{\langle x - y, b(x) - b(y) \rangle}{|x - y|} \leq -\frac{1}{|\Lambda|} |x - y|^3 + 2\beta |x - y|.$$

If we take  $f(r) = r$ , then for all  $x \neq y$ , we have

$$\bar{L}f(|x - y|) = \frac{\langle x - y, b(x) - b(y) \rangle}{|x - y|} f'(|x - y|) \leq -\left(\frac{1}{|\Lambda|} |x - y|^2 - 2\beta\right) |x - y|.$$

This is not enough for the exponential convergence except in the case that  $\beta < 0$  for which we have  $\inf_{r>0} (r^2/|\Lambda| - 2\beta) = -2\beta > 0$ . Due to this reason, we need a much carefully designed  $f$ . Define the function  $f$  as in the theorem, then we have

$$f'(r) = e^{-C(r)} \int_r^\infty e^C \sqrt{\varphi}, \quad f'' = -\frac{1}{4} \gamma f' - \sqrt{\varphi}.$$

We obtain

$$4f'' + \gamma f' = -4\sqrt{\varphi} \leq -\varepsilon f$$

with

$$\gamma(r) = -\frac{1}{|\Lambda|} r^3 + 2\beta r, \quad \varepsilon = 4 \inf_{r>0} \frac{\sqrt{\varphi(r)}}{f(r)}.$$

By the Cauchy mean value theorem, it follows that

$$\begin{aligned} \inf_{r>0} \frac{\sqrt{\varphi}}{f} &\geq \inf_{r>0} \frac{(\sqrt{\varphi})'}{f'} = \frac{1}{2} \inf_{r>0} \varphi^{-1/2} \Big/ \int_r^\infty e^C \sqrt{\varphi} \\ &\geq \frac{1}{2} \inf_{r>0} \frac{(\varphi^{-1/2})'}{-e^C \sqrt{\varphi}} = \frac{1}{4} \left( \inf_{r>0} \frac{e^{-C}}{\varphi} \right)^2 > 0. \end{aligned}$$

Therefore we obtain  $\varepsilon > 0$ . This proves our second assertion.

The exponential ergodicity is easy to check by using the so called “drift condition” with test function  $x \rightarrow |x|^2$ , but this is not enough to get a convergence rate. We now prove the exponential ergodicity with respect to  $f \circ |\cdot|$ . Note that here we do not assume that  $f \circ |\cdot|$  is a distance. Otherwise, the assertion follows from [17; Theorem 5.23]. We have proved in the last paragraph that  $\bar{\mathbb{E}}^{x,y} f(|X_t - Y_t|)$  is continuous in  $y$ . Moreover

$$\bar{\mathbb{E}}^{x,\mu_U} f(|X_t - Y_t|) = \int_{\mathbb{R}^{|\Lambda|}} \mu_U(dy) \bar{\mathbb{E}}^{x,y} f(|X_t - Y_t|) \leq e^{-\varepsilon t} \int_{\mathbb{R}^{|\Lambda|}} \mu_U(dy) f(|x - y|),$$

where  $\mu_U$  is the probability measure having density  $e^{-U}/Z_U$ , corresponding to the potential

$$U(x) = \sum_{i \in \Lambda} u(x_i) + J \sum_{i \in \Lambda} \sum_{j \in N(i)} (x_i - x_j)^2.$$

Because the left-hand side controls the Wasserstein distance, with respect to the cost function  $f \circ |\cdot|$ , of the laws of the processes starting from  $x$  and  $\mu_U$  respectively, we obtain an exponential ergodicity provided

$$\int_{\mathbb{R}^{|\Lambda|}} \mu_U(dy) f(|x - y|) < \infty.$$

To check this, noting that

$$-U(x) \leq \sum_{i \in \Lambda} (-x_i^4 + \beta x_i^2) \leq -\frac{1}{|\Lambda|} |x|^4 + \beta |x|^2$$

and  $f(|x - y|) \leq f(|x| + |y|)$ , it suffices to consider the radius part. That is,

$$\int_0^\infty f(r + z) \exp[-z^4/|\Lambda| + \beta z^2] dz < \infty \quad \text{for every } r \geq 0.$$

This can be done by using a comparison:

$$\begin{aligned} \frac{f(r + z) \exp[-z^4/|\Lambda| + \beta z^2]}{z^{-2}} &= \frac{f(r + z)}{z^{-2} \exp[z^4/|\Lambda| - \beta z^2]} \\ &\sim \frac{e^{-C(r+z)} \int_{r+z}^\infty e^C \sqrt{\varphi}}{[-2z^{-3} + z^{-2}(4z^3/|\Lambda| - 2\beta z)] \exp[z^4/|\Lambda| - \beta z^2]} \\ &\sim \frac{\int_{r+z}^\infty e^C \sqrt{\varphi}}{z \exp[z^4/|\Lambda| - \beta z^2 + C(r + z)]} \\ &\sim 0 \quad \text{as } z \rightarrow \infty. \end{aligned}$$

Finally, by [17; Theorem 9.18] and its remark, we also have  $\lambda_1(U, \Lambda, \beta) > 0$ .  $\square$

Theorem 6.1 is meaningful since it works for all finite dimensions. Note that  $\varepsilon(\Lambda, \beta) \rightarrow 0$  as  $|\Lambda| \rightarrow \infty$ , which is natural since the model exhibits a phase

transition. However, this result does not describe an ergodic region in the infinite dimensional situation.

For the remainder of this section, we apply the results obtained in the previous sections to some specific continuous spin systems. Denote by  $\langle ij \rangle$  the nearest bonds in  $\mathbb{Z}^d$ ,  $d \geq 1$ . Set  $N(i) = \{j : j \text{ is the endpoint of a bond } \langle ij \rangle\}$ . Then,  $|N(i)| :=$  the cardinality of the set  $N(i) = 2d$ . Consider the Hamiltonian  $H(x) = J \sum_{\langle ij \rangle} (x_i - x_j)^2$ , where  $J \geq 0$  is a constant. For a finite set  $\Lambda \subset \mathbb{Z}^d$  (denoted by  $\Lambda \Subset \mathbb{Z}^d$ ) and a point  $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ , define the finite-dimensional conditional Gibbs distribution  $\mu_U^{\Lambda, \omega}$  as follows.

$$\mu_U^{\Lambda, \omega}(dx_\Lambda) = e^{-U_\Lambda^\omega(x_\Lambda)} dx_\Lambda / Z_\Lambda^\omega, \quad (6.1)$$

where  $x_\Lambda = (x_i, i \in \Lambda)$ ,  $Z_\Lambda^\omega$  is the normalizing constant and

$$U_\Lambda^\omega(x_\Lambda) = \sum_{i \in \Lambda} u(x_i) + J \sum_{\langle ij \rangle: i, j \in \Lambda} (x_i - x_j)^2 + J \sum_{i \in \Lambda, j \in N(i) \setminus \Lambda} (x_i - \omega_j)^2 \quad (6.2)$$

for some function  $u \in C^\infty(\mathbb{R})$ , to be specified latterly. One can rewrite  $U_\Lambda^\omega$  as

$$U_\Lambda^\omega(x_\Lambda) = \sum_{i \in \Lambda} u(x_i) + J \sum_{i \in \Lambda} \sum_{j \in N(i)} (x_i - z_j)^2, \quad (6.3)$$

where

$$z_j = \begin{cases} x_j, & \text{if } j \in \Lambda \\ \omega_j, & \text{if } j \notin \Lambda. \end{cases}$$

Correspondingly, we have an operator  $L_\Lambda^\omega$  and a Dirichlet form  $D_\Lambda^\omega$  as follows.

$$L_\Lambda^\omega = \Delta_\Lambda - \langle \nabla_\Lambda U_\Lambda^\omega, \nabla_\Lambda \rangle, \quad D_\Lambda^\omega(f) = \int_{\mathbb{R}^{|\Lambda|}} |\nabla_\Lambda f|^2 d\mu_U^{\Lambda, \omega}. \quad (6.4)$$

Our purpose in this section is to estimate  $\lambda_1(L_\Lambda^\omega) = \lambda_1(U_\Lambda^\omega)$ . By (1.6), we have the simplest lower bound of the marginal eigenvalues as follows.

$$\lambda_1^{x_{\Lambda \setminus i}, \omega} \geq \inf_{x \in \mathbb{R}} u''(x) + 4dJ, \quad (6.5)$$

where  $x_{\Lambda \setminus i} = (x_j, j \in \Lambda \setminus \{i\})$ . The function  $C(x)$  defined in Section 4 becomes

$$\begin{aligned} C_\Lambda^{x_{\Lambda \setminus i}, \omega}(x_i) &= -u(x_i) - J \sum_{j \in N(i)} (x_i - z_j)^2 \\ &= -u(x_i) - 2dJx_i^2 + 2J \left( \sum_{j \in N(i)} z_j \right) x_i - J \sum_{j \in N(i)} z_j^2, \\ &\quad i \in \Lambda. \end{aligned} \quad (6.6)$$

The last term can be ignored, since it does not make influence to  $\mu_U^{x_{\Lambda \setminus i}}$ , and so neither  $\lambda_1^{x_{\Lambda \setminus i}}$ . The coefficient of the second to the last term varies over whole  $\mathbb{R}$  if  $J \neq 0$ .

We consider two models only:  $u(x) = \alpha x^2$  and  $u(x) = x^4 - \beta x^2$  for some constants  $\alpha > 0$  and  $\beta \in \mathbb{R}$ , respectively.

**Theorem 6.2.** *Let  $u(x) = \alpha x^2$  for some constant  $\alpha > 0$  and let  $U(x) = \sum_i u(x_i) + H(x)$  with Hamiltonian  $H(x) = J \sum_{\langle ij \rangle} (x_i - x_j)^2$ . Then we have*

$$\inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{Z^d}} \lambda_1(U_\Lambda^\omega) \geq \inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{Z^d}} \sigma(U_\Lambda^\omega) \geq 2\alpha. \quad (6.7)$$

*Proof.* It suffices to prove the second estimate. By Example 5.2 and Theorem 1.3, the proof is very much the same as proving

$$\inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{Z^d}} \lambda_1(U_\Lambda^\omega) \geq 2\alpha.$$

Hence we prove here the last assertion only. First, we have

$$|\partial_{ij}U(x)| = \begin{cases} 2J, & i, j \in \Lambda, |i - j| = 1 \\ 0, & i, j \in \Lambda, |i - j| > 1. \end{cases} \quad (6.8)$$

The right-hand side is independent of  $x$ , which is the main reason why we were looking for the uniform estimates (with respect to the linear term) in Examples 4.2 and 4.3. By (6.5), we have  $\lambda_1^{x_{\Lambda \setminus i}, \omega} \geq 2\alpha + 4dJ$ , which is indeed sharp in view of Example 4.2. Combining these facts together and using (1.4) with  $w_i \equiv 1$ , it follows that

$$\begin{aligned} \lambda_1(U_\Lambda^\omega) &\geq \inf_{x \in \mathbb{R}^{|\Lambda|}} \min_{i \in \Lambda} \left[ 2\alpha + 4dJ - \sum_{j \in \Lambda: |i-j|=1} 2J \right] \\ &= 2\alpha + 4dJ - 2J \max_{i \in \Lambda} |\{ \langle i, j \rangle : j \in \Lambda \}| \\ &\geq 2\alpha \end{aligned}$$

uniformly in  $\omega \in \mathbb{R}^{Z^d}$  and  $\Lambda \in \mathbb{Z}^d$ . The sign of the last equality holds once  $\Lambda$  contains a point together with all of its neighbors.  $\square$

In the last step of the proof, we did not use Theorem 1.2 since the matrix  $(|\partial_{ij}U(x)| : i, j \in \Lambda)$  is very simple. Nevertheless, it provides us a good chance to justify the power of Theorem 1.2. To do so, take  $\eta_i^{x_{\Lambda \setminus i}} = 2\alpha + 4dJ = \lambda_1^{x_{\Lambda \setminus i}}$ . Then

$$\begin{aligned} s_i(x) &= \eta_i^{x_{\Lambda \setminus i}} - \sum_{j \in \Lambda: j \neq i} |\partial_{ij}U(x)| = 2\alpha + 4dJ - 2J |\{ \langle i, j \rangle : j \in \Lambda \}|, \quad i \in \Lambda, \\ \underline{s}(x) &= \min_{i \in \Lambda} s_i(x) = 2\alpha. \end{aligned}$$

Since  $h^{(\gamma)} \geq 0$ , Theorem 1.2 already gives us  $\lambda_1(U_\Lambda^\omega) \geq \inf_x \underline{s}(x) = 2\alpha$  as expected, without using  $h^{(\gamma)}$ . To see the role played by  $h^{(\gamma)}$ , note that

$$\begin{aligned} q_i(x) &= \eta_i^{x_{\Lambda \setminus i}} - \underline{s}(x) = 4dJ, \quad i \in \Lambda, \\ d_i(x) &= s_i(x) - \underline{s}(x) = 4dJ - 2J |\{ \langle i, j \rangle : j \in \Lambda \}|, \quad i \in \Lambda. \end{aligned}$$

Note that  $d_i(x)$  here depends on  $i$ . Thus

$$\begin{aligned}
h^{(\gamma)}(x) &= \min_{A: \emptyset \neq A \subset \Lambda} \frac{1}{|A|} \left[ \sum_{i \in A} \frac{d_i(x)}{q_i(x)^\gamma} + \sum_{i \in A, j \in \Lambda \setminus A} \frac{|\partial_{ij} U(x)|}{[q_i(x) \vee q_j(x)]^\gamma} \right] \\
&= \frac{2J}{(4dJ)^\gamma} \min_{A: \emptyset \neq A \subset \Lambda} \frac{1}{|A|} \sum_{i \in A} \left[ 2d - |\{\langle i, j \rangle : j \in \Lambda\}| + |\{\langle i, j \rangle : j \in \Lambda \setminus A\}| \right] \\
&= \frac{2J}{(4dJ)^\gamma} \min_{A: \emptyset \neq A \subset \Lambda} \frac{1}{|A|} \sum_{i \in A} \left[ |\{\langle i, j \rangle\}| - |\{\langle i, j \rangle : j \in A\}| \right] \\
&= \frac{2J}{(4dJ)^\gamma} \min_{A: \emptyset \neq A \subset \Lambda} \frac{1}{|A|} \sum_{i \in A} |\{\langle i, j \rangle : j \notin A\}|. \\
&=: \frac{2J}{(4dJ)^\gamma} \min_{A: \emptyset \neq A \subset \Lambda} \frac{|\partial A|}{|A|}.
\end{aligned}$$

Clearly, the right-hand side depends reasonably on the geometry of  $\Lambda$ . Roughly speaking, by the isoperimetric principle, the last minimum of the ratio is approximately  $|\partial B|/|B|$ , where  $B$  is the largest ball contained in  $\Lambda$ . Anyhow, for regular  $\Lambda$  (cube for instance),

$$h^{(\gamma)}(x) \leq \frac{2J}{(4dJ)^\gamma} \cdot \frac{|\partial \Lambda|}{|\Lambda|} \rightarrow 0 \quad \text{as } \Lambda \uparrow \mathbb{Z}^d.$$

Hence for this model,  $h^{(\gamma)}$  makes no contribution to  $\lambda_1(U_\Lambda^\omega)$  for the estimate uniformly in  $\Lambda$ .

**Theorem 6.3.** *Let  $u(x) = x^4 - \beta x^2$  for some constant  $\beta \in \mathbb{R}$  and let  $U(x) = \sum_i u(x_i) + H(x)$  with Hamiltonian  $H(x) = -2J \sum_{\langle ij \rangle} x_i x_j$ . Then we have*

$$\begin{aligned}
\inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \lambda_1(U_\Lambda^\omega) &\geq \inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \sigma(U_\Lambda^\omega) \\
&\geq \frac{\sqrt{\beta^2 + 8} - \beta}{\sqrt{e}} \exp \left[ -\frac{1}{8} \beta (\beta + \sqrt{\beta^2 + 8}) \right] - 4dJ, \quad (6.9)
\end{aligned}$$

For simplicity, we write  $r = 2dJ$ . The right-hand side is positive if  $(\beta, r) \in \mathbb{R} \times \mathbb{R}_+$  is located in the region below the curve in Figure 1 (including the region of  $\beta \leq 0$  vertically below the shade one.)

*Proof.* As shown in part (2) of Example 4.6, for zero boundary condition  $\omega = 0$ , we have

$$\lim_{\beta \rightarrow \infty} \sigma^{x_{\Lambda \setminus i}, \omega} \leq \lim_{\beta \rightarrow \infty} \lambda_1^{x_{\Lambda \setminus i}, \omega} = 0.$$

In other words, due to the double-well potential, the spectral gap and then the logarithmic constant will be absorbed as  $\beta \rightarrow \infty$ . Combining Example 5.3 with Theorem 1.3 and following the last step of the proof Theorem 6.2, we obtain the required lower estimate.  $\square$

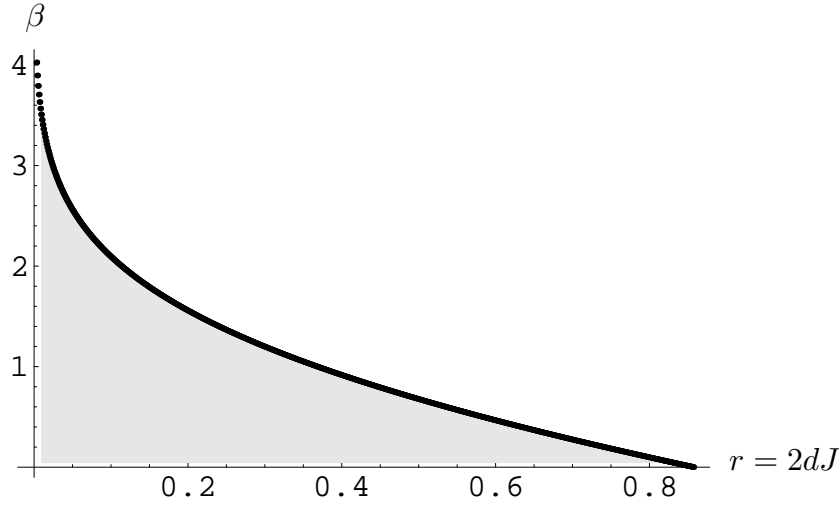


Figure 1.

For the Hamiltonian  $H(x) = J \sum_{\langle ij \rangle} (x_i - x_j)^2$  discussed several times before, simply replacing  $\beta$  with  $\beta - 2dJ$  in Theorem 6.3, we obtain the following estimate:

$$\begin{aligned}
 & \inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \lambda_1(U_\Lambda^\omega) \\
 & \geq \inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \sigma(U_\Lambda^\omega) \\
 & \geq \frac{\sqrt{(\beta - r)^2 + 8} - \beta + r}{\sqrt{e}} \exp \left[ -\frac{1}{8}(\beta - r) \left( \beta - r + \sqrt{(\beta - r)^2 + 8} \right) \right] \\
 & \quad - 2r, \tag{6.10}
 \end{aligned}$$

where  $r = 2dJ$ . The ergodic region is shown in Figure 2.

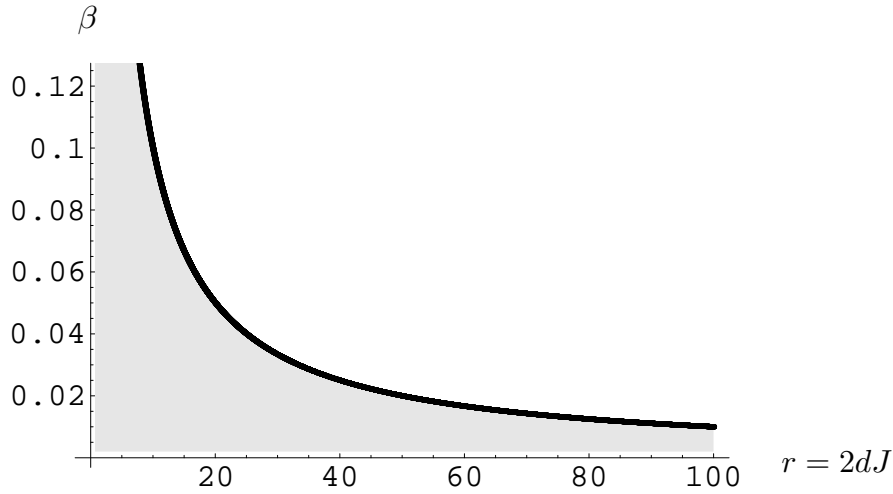


Figure 2.

**Remark 6.4.** As mentioned below the proof of Proposition 3.1, by considering the interacting terms more carefully, one may improve Theorem 1.1 for stronger interactions. For instance, since the variance of a random variable having the distribution with density  $\exp[-x^4 + \beta x^2]/Z$  on the real line is asymptotically  $\beta/2$  for  $\beta \geq 0$ , and is bounded above by

$$\frac{\Gamma(3/4)}{\Gamma(1/4)} + \beta \left/ \left[ 2 + \frac{4\Gamma(1/4)}{9(1+\beta)\Gamma(3/4)} \right] \right.,$$

by using [9; Proposition 5.8], when  $\beta \geq 0$ , the lower bound of  $\inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \lambda_1(U_\Lambda^\omega)$  given in Theorem 6.3 can be improved as follows: replacing the interaction term  $4dJ$  in (6.9) with

$$4dJ \left[ \frac{\Gamma(3/4)}{\Gamma(1/4)} + \beta \left/ \left[ 2 + \frac{4\Gamma(1/4)}{9(1+\beta)\Gamma(3/4)} \right] \right. \right] \frac{\sqrt{\beta^2 + 8} - \beta}{\sqrt{e}} \exp \left[ -\frac{1}{8}\beta(\beta + \sqrt{\beta^2 + 8}) \right]. \quad (6.11)$$

Finally, we mention that there is another technique which works even in the irreversible situation (cf. [17; Theorem 14.10]) to handle with the exponentially ergodic region, because the second model (Theorem 6.3) is attractive (stochastic monotone) and has the moments of all orders, plus a use of the translation invariant. However, as known that the logarithmic Sobolev inequality already implies an exponential ergodicity in the entropy and moreover, the usual exponential ergodicity is equivalent to the Poincaré inequality with nearly the same convergence exponent in the present context (cf. [13; Theorem 8.13]), there is almost no room to improve the ergodic region.

**Acknowledgements.** This paper takes an unusual long period in preparation. The most part of the paper was finished in 2002, but the exact coefficient  $1/4$  of decay rate  $\exp[-\beta^2/4]$  for the second model (Theorem 6.3) was left to be open and so the earlier draft was communicated within a small group only. Recently, the author came back to compute the logarithmic Sobolev constant which leads to the precise coefficient and hence completes the paper. The author would like to acknowledge the organizers for several conferences in which partial results of the paper were presented: I. Shigekawa (the 11th International Research Institute of Mathematical Society of Japan, 2002), L.M. Wu (Chinese and French Workshop on Probability and Applications, 2004), Z.M. Ma and M. Röckner (Second Sino-German Meeting on Stochastic Analysis, 2007). The author is also greatly indebted to the referees for their helpful comments.

## REFERENCES

- [1]. Bodineau, T. and Helffer, B. (1999), *The log-Sobolev inequality for unbounded spins systems*, J. Funct. Anal. 166:1, 168-178.
- [2]. Bodineau, T. and Helffer, B. (2000), *Correlations, spectral gap and log-Sobolev inequalities for unbounded spins systems*, Diff. Eq. Math. Phys., AMS/IP Stud. Adv. Math. 16, Amer. Math. Soc., Providence, RI, 51-66.

- [3]. Deuschel J.-D. and D. W. Stroock (1990), *Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models*, J. Funct. Anal. 92, 30–48.
- [4]. Gao, F.Q. and Wu, L.M. (2007), *Transportation-information inequalities for Gibbs measures*, preprint.
- [5]. Gentil, I. and Roberto, C. (2001), *Spectral gaps for spin systems: some non-convex phase examples*, J. Funct. Anal. 180, 66–84.
- [6]. Helffer, B. (1999), *Remarks on decay of correlations and Witten Laplacians III. Application to logarithmic Sobolev inequalities*, Ann. Inst. H. Poincaré (B), Prob. Stat. 35:4, 483–508.
- [7]. Ledoux, M. (2001), *Logarithmic Sobolev inequalities for unbounded spin systems revised*, “Séminaire de Probabilités” XXXV. LNM 1755, 167–194. Springer.
- [8]. Otto, F. and Reznikoff, M.G. (2007), *A new criterion for the logarithmic Sobolev inequality and two applications*, J. Funct. Anal. 243, 121–157.
- [9]. Wu, L. M. (2006), *Poincaré and transportation inequalities for Gibbs measures under the Dobrushin uniqueness condition*, Ann. Prob. 34:5, 1960–1989.
- [10]. Yoshida, N. (1999), *The log-Sobolev inequality for weakly coupled lattice field*, Prob. Theor. Relat. Fields 115, 1–40.
- [11]. Zegarlinski, B. (1996), *The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice*, Comm. Math. Phys. 175, 401–432.
- [12]. Chen, M. F. and Wang, F. Y. (1997), *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. 349, 1239–1267.
- [13]. Chen, M. F. (2005), *Eigenvalues, Inequalities, and Ergodic Theory*, Springer, London.
- [14]. Bakry, D. and Emery, M. (1985), *Diffusions hypercontractives*, in “Séminaire de probabilités” XIX, 1983/84, Springer, Berlin, 177–206.
- [15]. Chen, M. F. (2000), *The principal eigenvalue for jump processes*, Acta Math. Sin. Eng. Ser. 16:3, 361–368.
- [16]. Chen, M. F. and Wang, F. Y. (1998), *Cheeger’s inequalities for general symmetric forms and existence criteria for spectral gap*, Abstract. Chin. Sci. Bull. 43:18, 1516–1519. Ann. Prob. 2000, 28:1, 235–257.
- [17]. Chen, M. F. (1992), *From Markov Chains to Non-Equilibrium Particle Systems.*, World Scientific. 2nd ed. 2004.
- [18]. Chen, M. F. (2000), *Explicit bounds of the first eigenvalue*, Sci. Chin. Ser. A 43(10), 1051–1059.
- [19]. Chen, M. F. (2001), *Variational formulas and approximation theorems for the first eigenvalue in dimension one*, Sci. Chin. Ser. A 44(4), 409–418.
- [20]. Conte, J. M., et al (1963/64), *Solution of Problem 5607*, Revue Math. Spéc. 74, 227–230.
- [21]. Gautschi, W. (1959), *Some elementary inequalities relating to the gamma and incomplete gammafunction*, J. Math. and Phys. 38, 77–81.
- [22]. Chen, M. F. and Wang, F. Y. (1997), *General formula for lower bound of the first eigenvalue on Riemannian manifolds*, Sci. Chin. Ser. A 40:4, 384–394.
- [23]. Chen, M. F. and F. Y. Wang (1997), *Estimates of logarithmic Sobolev constant*, J. Funct. Anal. 144:2, 287–300.

Note: Figures 1 and 2 were missed in the publication and the correction appeared in the same journal, Vol. 25, No. 12, pp. 2199–2199.